ORDINARY DIFFERENTIAL EQUATIONS

BAMAT-201

Self Learning Material

Directorate of Distance Education

SWAMI VIVEKANAND SUBHARTI UNIVERSITY

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UTTAR PRADESH

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SEMESTER II

Course I

Course Name: Ordinary Differential Equations (ODE) Course Code: BAMAT-201

Course Objectives:	The main objectives of this course are to introduce the students to the exciting world of Differential Equations and their applications.
Unit 1:	Formation of differential equation, Degree, order and solution of a D.E., Ordinary differential equations of first order: initial and boundary conditions,Seperation of variables method, homogeneous equations:equation reducible to Homogeneous Form, linear equations, Equation reducible to homogeneous form
Unit 2:	Exact differential Equation. Necessary and sufficient condition for exact differential equation, First order higher degree equations solvable for x, y, p. Singular solution and envelopes,Clairaut's equation,Equation Reducible to Clauriat,s form.
Unit 3:	Linear differential equations with constant coefficients; Determinaton of C.F. and the P.I., homogeneous linear differential equations, Determinaton of C.F. and the P.I., linear differential equations of second order with variable coefficients,
Unit 4:	Series solutions of differential equations. Introduction Frobenious Method Solution near an ordinary point and a regular singular point, Method of differentiation, Bessel and Legendre equations. Solution of Legendre equation, Defination of Legendre polynomials, Bessel and Legendre functions.

Course Learning Outcomes: The course will enable the students to:

1. Formulate Differential Equations for various Mathematical models.
2. Solve first order non-linear differential equation and linear differential equations of higher order using various techniques.
3. Apply these techniques to solve and analyze various mathematical models.

References:

1. Barnes, Belinda & Fulford, Glenn R. (2015). *Mathematical Modelling with Case Studies, Using* Maple *and* MATLAB (3rd ed.). CRC Press, Taylor & Francis Group.
2. Edwards, C. Henry, Penney, David E., & Calvis, David T. (2015). *Differential Equation and Boundary Value Problems*: *Computing and Modeling* (5th ed.). Pearson Education.

Ross, Shepley L. (2004). *Differential Equations* (3rd ed.). John Wiley & Sons. India

UNIT I

1. DIFFERENTIAL EQUATIONS

NOTES

STRUCTURE

Differential Equation
Formation of a Differential Equation Whose General Solution is Given
Solution of a Differential Equation
Initial Value Problem
Solution of a Differential Equation by the Method of Separation of Variables
Homogeneous Differential Equations and their Solution

DIFFERENTIAL EQUATION

An equation involving independent variables, dependent variables and at least one derivative/differential of these variables is called a **differential equation**.

The following are some of the examples of differential equations:

1. $\frac{dy}{dx} = x \log x$
2. $dy = \cos x \, dx$
3. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 12y = x^4$
4. $y = x\frac{dy}{dx} + a\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$.

Order and Degree of a Differential Equation

The **order** of a differential equation is the order of the derivative of the highest order, occurring in the differential equation.

Consider the differential equation

$$3\frac{d^3y}{dx^3} + \frac{dy}{dx} + y = \sin x.$$

The highest order derivative occurring in this equation is $\frac{d^3y}{dx^3}$ and its order is 3.

NOTES

$\therefore$ Order of given differential equation is 3.

The **degree** of a differential equation is defined if it can be written as a polynomial equation in the derivatives and for such a differential equation its degree is given by the highest power of the highest order derivative appearing in it, provided the derivatives are made free from radicals and fractions.

Consider the differential equation

$$y = 2\frac{dy}{dx} + 3\sqrt{1+2\left(\frac{dy}{dx}\right)^2}. \qquad ...(1)$$

This equation is not free from radicals.

$$(1) \Rightarrow \qquad y - 2\frac{dy}{dx} = 3\sqrt{1+2\left(\frac{dy}{dx}\right)^2}$$

$$\Rightarrow \qquad \left(y - 2\frac{dy}{dx}\right)^2 = 9\left(1+2\left(\frac{dy}{dx}\right)^2\right)$$

$$\Rightarrow \qquad y^2 + 4\left(\frac{dy}{dx}\right)^2 - 4y\frac{dy}{dx} - 9 - 18\left(\frac{dy}{dx}\right)^2 = 0$$

$$\Rightarrow \qquad 14\left(\frac{dy}{dx}\right)^2 + 4y\frac{dy}{dx} - y^2 + 9 = 0$$

The highest order derivative in this equation is $\frac{dy}{dx}$ and its highest power is 2.

$\therefore$ Degree of given differential equation is 2.

Linear Differential Equation

A differential equation is said to be **linear**, if the dependent variable and its derivatives occur only in the first degree and are not multiplied together.

In general, a linear differential equation of order n is of the form

$$P_0\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + \dots + P_{n-1}\frac{dy}{dx} + P_ny = Q,$$

where $P_0, P_1, \dots, P_{n-1}, P_n$, Q are functions of x or constants.

In particular, a linear differential equation of order *one* is of the form

$$P_0\frac{dy}{dx} + P_1y = Q.$$

The differential equations: $\cos x\,\frac{dy}{dx} + y\sin x = 1$ and $\frac{d^3y}{dx^3} + \frac{y}{x} = x^2\log x$ are linear differential equations.

A differential equation which is not linear is called **non-linear**.

The degree of a linear differential equation is always one. But, the converse is not true. For example, the degree of $y\frac{dy}{dx} + 7 = \sin x$ is one and it is not a linear differential equation.

SOLVED EXAMPLES

NOTES

Example 1. *Determine the order and degree, if defined, of the following differential equations. State also, if these are linear or non-linear:*

(*i*) $xy\dfrac{dy}{dx} = \dfrac{(1+y^2)(1+x+x^2)}{1+x^2}$ (*ii*) $y = \dfrac{dy}{dx} + \sqrt{1+\left(\dfrac{dy}{dx}\right)^3}$.

Solution. (*i*) The given differential equation is $xy\dfrac{dy}{dx} = \dfrac{(1+y^2)(1+x+x^2)}{1+x^2}$.

Order of the highest order derivative $\dfrac{dy}{dx}$ is 1.

Highest power of the highest order derivative $\dfrac{dy}{dx}$ is 1.

∴ Order and degree of the given differential equation are **1** each.

The given differential equation is **non-linear,** because y and $\dfrac{dy}{dx}$ are multiplied together.

(*ii*) The given differential equation is $y = \dfrac{dy}{dx} + \sqrt{1+\left(\dfrac{dy}{dx}\right)^3}$. ...(1)

Order of the highest order derivative $\dfrac{dy}{dx}$ is 1.

(1) $\Rightarrow$ $y - \dfrac{dy}{dx} = \sqrt{1+\left(\dfrac{dy}{dx}\right)^3}$ $\Rightarrow$ $\left(y - \dfrac{dy}{dx}\right)^2 = 1 + \left(\dfrac{dy}{dx}\right)^3$

This is expressible as a polynomial in $\dfrac{dy}{dx}$.

Highest power of the highest order derivative $\dfrac{dy}{dx}$ is 3.

∴ Order and degree of the given differential equation are **1** and **3** respectively.

The given differential equation is **non-linear** because $\dfrac{dy}{dx}$ is multiplied by itself.

EXERCISE A

Determine the order and degree, if defined, of the following differential equations. State also, if these are linear or non-linear (Q No. 1–4):

1. (*i*) $x^3\left(\dfrac{d^2y}{dx^2}\right)^2 + x\left(\dfrac{dy}{dx}\right)^4 = 0$ (*ii*) $\left(\dfrac{dy}{dx}\right)^4 + 3x\dfrac{d^2y}{dx^2} = 0$

(*iii*) $\left(\dfrac{dy}{dx}\right)^4 + 3y\dfrac{d^2y}{dx^2} = 0$ (*iv*) $\left(\dfrac{d^2y}{dx^2}\right)^3 + y\left(\dfrac{dy}{dx}\right)^4 + x^3 = 0$

2. (*i*) $5x\left(\dfrac{dy}{dx}\right)^2 + \dfrac{d^2y}{dx^2} - 6y = \log x$ (*ii*) $y\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^3 = x\left(\dfrac{d^3y}{dx^3}\right)^2$

(*iii*) $y''^2 - 2y'' - y' + 1 = 0$ (*iv*) $y = x\dfrac{dy}{dx} + a\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$

3. (*i*) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = 5\,\frac{d^2y}{dx^2}$ (*ii*) $\sqrt{1-x^2}\ dx + \sqrt{1-y^2}\ dy = 0$

(*iii*) $\left(\frac{ds}{dt}\right)^4 + 3s\,\frac{d^2s}{dt^2} = 0$ (*iv*) $y = px + \sqrt{a^2p^2+b^2}$, where $p = \frac{dy}{dx}$

4. (*i*) $\frac{d^4y}{dx^4} + \sin(y''') = 0$ (*ii*) $\left(\frac{d^2y}{dx^2}\right)^2 + \cos\left(\frac{dy}{dx}\right) = 0$

(*iii*) $(y''')^2 + (y'')^3 + (y')^4 + y^5 = 0$ (*iv*) $y'' + (y')^2 + 2y = 0.$

5. Write the sum of the order and degree of the following differential equations:

(*i*) $\frac{d}{dx}\left\{\left(\frac{dy}{dx}\right)^3\right\} = 0$ (*ii*) $\frac{d^2y}{dx^2} + \sqrt[3]{\frac{dy}{dx}} + (1+x) = 0.$

Answers

1. (*i*) Order = 2, degree = 2, non-linear (*ii*) Order = 2, degree = 1, non-linear

(*iii*) Order = 2, degree = 1, non-linear (*iv*) Order = 2, degree = 3, non-linear

2. (*i*) Order = 2, degree = 1, non-linear (*ii*) Order = 3, degree = 2, non-linear

(*iii*) Order = 2, degree = 2, non-linear (*iv*) Order = 1, degree = 2, non-linear

3. (*i*) Order = 2, degree = 2, non-linear (*ii*) Order = 1, degree = 1, non-linear

(*iii*) Order = 2, degree = 1, non-linear (*iv*) Order = 1, degree = 2, non-linear

4. (*i*) Order = 4, degree not defined, non-linear

(*ii*) Order = 2, degree not defined, non-linear

(*iii*) Order = 3, degree = 2, non-linear

(*iv*) Order = 2, degree = 1, non-linear

5. (*i*) 3 (*ii*) 5.

FORMATION OF A DIFFERENTIAL EQUATION WHOSE GENERAL SOLUTION IS GIVEN

If we have an equation between two variables, involving arbitrary constants, then these arbitrary constants can be eliminated by using derivatives and as a result, a differential equation is formed whose solution is the given equation.

I. Method of Forming a Differential Equation

To form a differential equation from a given equation in *x, y* and containing arbitrary constants, the given equation is differentiated w.r.t. *x* successively as many times as there are arbitrary constants. These equations are used to eliminate the arbitrary constants. The equation obtained by eliminating the arbitrary constants is the required differential equation.

In general, if the equation between two variables contains n arbitrary constants, then the differential equation, obtained by eliminating these arbitrary constants, will be of order n.

NOTES

Remark. The following are some of the important results of coordinate geometry which are used in this section.

1. Equation of non-vertical line is $y = mx + c$, where m and c are arbitrary constants.
2. Equation of a non-vertical line passing through the origin is $y = mx$, where m is arbitrary constant.
3. Equation of the circle having centre (h, k) and radius r is $(x - h)^2 + (y - k)^2 = r^2$.
4. Equation of circle in the general form is $x^2 + y^2 + 2gx + 2fy + c = 0$. Its centre and radius are $(-g, -f)$ and $\sqrt{g^2 + f^2 - c}$ respectively.
5. Equation of a circle passing through the origin is $x^2 + y^2 + 2gx + 2fy = 0$, where g and f are arbitrary constants.
6. Equation of a circle passing through the origin and having centre on the x-axis is $(x - a)^2 + y^2 = a^2$, where a is arbitrary constant.
7. Equation of a circle passing through the origin and having centre on the y-axis is $x^2 + (y - a)^2 = a^2$, where a is arbitrary constant.
8. Equation of a parabola with axis parallel to the x-axis is $(y - k)^2 = 4a(x - h)$, where a, h and k are arbitrary constants.
9. Equation of a parabola with axis parallel to the y-axis is $(x - h)^2 = 4a(y - k)$, where a, h and k are arbitrary constants.
10. Equation of an ellipse having centre at the origin and axes along the coordinate axes is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a and b are arbitrary constants.

Working Steps for the Formation of Differential Equations

Step I. Write the given equation.

Step II. Count the number of distinct arbitrary constants present in the given equation.

Step III. Differentiate the given equation successively as many times as the number of arbitrary constants.

Step IV. Eliminate the arbitrary constants by using the given equation and equations obtained in the ***step III***. The equation so obtained is the required differential equation.

SOLVED EXAMPLES

Example 2. *Form the differential equation of the following families of curves:*

(*i*) *$y = mx$, where m is an arbitrary constant.*

(*ii*) *$(x - a)^2 + 2y^2 = a^2$, where a is an arbitrary constant.*

Solution. (*i*) We have $y = mx$. ...(1)

Differentiating (1) w.r.t. x, we get $\frac{dy}{dx} = m$

Elimination of *m*. Putting the value of m in (1), we get $y = \left(\frac{dy}{dx}\right) x$.

This is the required differential equation.

(*ii*) We have $(x - a)^2 + 2y^2 = a^2$ *i.e.,* $x^2 - 2ax + 2y^2 = 0$ ...(1)

Differentiating (1) $\Rightarrow$ $2x - 2a + 4yy' = 0$ $\Rightarrow$ $a = x + 2yy'$

NOTES

Elimination of *a*. Putting $a = x + 2yy'$ in (1), we get

$$x^2 - 2(x + 2yy')x + 2y^2 = 0$$

$$\Rightarrow \quad x^2 - 2x^2 - 4xyy' + 2y^2 = 0 \quad \Rightarrow \quad 4xyy' + x^2 - 2y^2 = 0$$

$\Rightarrow \quad \mathbf{4xy \dfrac{dy}{dx} + x^2 - 2y^2 = 0}$. This is the required differential equation.

> **Remark.** The differential equation obtained for each system in the above example is of order 'one'. This is so, because each system contained only one arbitrary constant.

Example 3. *Form the differential equation of the following families of curves:*

(*i*) $y = Ax + \dfrac{B}{x}$, *where A, B are arbitrary constants.*

(*ii*) $y = Ae^{3x} + Be^{5x}$, *where A, B are arbitrary constants.*

Solution (*i*) We have $\quad y = Ax + \dfrac{B}{x}$.

$$\Rightarrow \quad xy = Ax^2 + B \qquad \text{...(1)}$$

Differentiating (1) w.r.t. x, we get $\quad x\dfrac{dy}{dx} + y \cdot 1 = A \cdot 2x + 0$

$$\Rightarrow \quad x\frac{dy}{dx} + y = 2Ax \qquad \text{...(2)}$$

Differentiating (2) w.r.t. x, we get $\quad \left(x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} \cdot 1\right) + \dfrac{dy}{dx} = 2A \cdot 1.$

$$\Rightarrow \quad x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 2A$$

Elimination of A and B. Putting the value of 2A in (2), we get

$$x\frac{dy}{dx} + y = \left(x\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\right)x$$

$\Rightarrow \quad \mathbf{x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = 0}$. This is the required differential equation.

(*ii*) We have $\quad y = Ae^{3x} + Be^{5x}$. $\qquad$ ...(1)

Differentiating (1) w.r.t. x, we get $\quad \dfrac{dy}{dx} = 3Ae^{3x} + 5Be^{5x}$ $\qquad$ ...(2)

Differentiating again $\quad \dfrac{d^2y}{dx^2} = 9Ae^{3x} + 25Be^{5x}$ $\qquad$ ...(3)

Elimination of A and B.

$$(3) - 5(2) \quad \Rightarrow \quad \frac{d^2y}{dx^2} - 5\frac{dy}{dx} = -6Ae^{3x} \quad \Rightarrow \quad -\frac{1}{6}\frac{d^2y}{dx^2} + \frac{5}{6}\frac{dy}{dx} = Ae^{3x}$$

$$(3) - 3(2) \quad \Rightarrow \quad \frac{d^2y}{dx^2} - 3\frac{dy}{dx} = 10Be^{5x} \quad \Rightarrow \quad \frac{1}{10}\frac{d^2y}{dx^2} - \frac{3}{10}\frac{dy}{dx} = Be^{5x}$$

$$\therefore \ (1) \quad \Rightarrow \quad y = \left(-\frac{1}{6}\frac{d^2y}{dx^2} + \frac{5}{6}\frac{dy}{dx}\right) + \left(\frac{1}{10}\frac{d^2y}{dx^2} - \frac{3}{10}\frac{dy}{dx}\right)$$

$$\Rightarrow \quad 30y = -5\frac{d^2y}{dx^2} + 25\frac{dy}{dx} + 3\frac{d^2y}{dx^2} - 9\frac{dy}{dx}$$

NOTES

$$\Rightarrow \qquad 2\frac{d^2y}{dx^2} - 16\frac{dy}{dx} + 30y = 0$$

$\Rightarrow$ $\boldsymbol{\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0}$. This is the required differential equation.

Alternative Method

We have $\qquad y = Ae^{3x} + Be^{5x}$. ...(1)

(1) $\Rightarrow$ $\qquad y_1 = 3Ae^{3x} + 5Be^{5x}$ ...(2)

(2) – 3(1) $\Rightarrow$ $\qquad y_1 - 3y = 2Be^{5x}$ ...(3)

(3) $\Rightarrow$ $y_2 - 3y_1 = 10Be^{5x}$ $\Rightarrow$ $y_2 - 3y_1 = 5(y_1 - 3y)$ $\Rightarrow$ $y_2 - 8y_1 + 15y = 0$

$\Rightarrow$ $\boldsymbol{\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0}$. This is the required differential equation.

> **Remark** The differential equation obtained for each system in the above example is of order 'two'. This is so, because each system contained two arbitrary constants.

Example 4. *Form the differential equation of the following families of curves:*

(i) $y = ae^x + be^{2x} + ce^{-3x}$, *where a, b, c are arbitrary constants.*

(ii) $x^2 + y^2 + 2ax + 2by + c = 0$, *where a, b, c are arbitrary constants.*

Solution *(i)* We have $\qquad y = ae^x + be^{2x} + ce^{-3x}$. ...(1)

(1) $\Rightarrow$ $\qquad y_1 = ae^x + 2be^{2x} - 3ce^{-3x}$ ...(2)

(2) – (1) $\Rightarrow$ $\qquad y_1 - y = be^{2x} - 4ce^{-3x}$ ...(3)

(3) $\Rightarrow$ $\qquad y_2 - y_1 = 2be^{2x} + 12ce^{-3x}$ ...(4)

(4) – 2(3) $\Rightarrow$ $\qquad y_2 - y_1 - 2(y_1 - y) = 20ce^{-3x}$

$\Rightarrow$ $\qquad y_2 - 3y_1 + 2y = 20ce^{-3x}$ ...(5)

(5) $\Rightarrow$ $\qquad y_3 - 3y_2 + 2y_1 = -60ce^{-3x}$ ...(6)

(6) + 3(5) $\Rightarrow$ $\qquad y_3 - 3y_2 + 2y_1 + 3(y_2 - 3y_1 + 2y) = 0$

$\Rightarrow$ $\qquad y_3 - 7y_1 + 6y = 0$

$\Rightarrow$ $\boldsymbol{\frac{d^3y}{dx^3} - 7\frac{dy}{dx} + 6y = 0}$. This is the required differential equation.

(ii) We have $\qquad x^2 + y^2 + 2ax + 2by + c = 0$. ...(1)

Differentiating (1) w.r.t. x, we get $\qquad 2x + 2yy_1 + 2a + 2by_1 + 0 = 0$.

$\Rightarrow$ $\qquad x + yy_1 + a + by_1 = 0$ ...(2)

Differentiating (2) w.r.t. x, we get $\qquad 1 + (yy_2 + y_1y_1) + 0 + by_2 = 0$

$\Rightarrow$ $\qquad by_2 = -(1 + yy_2 + y_1^2)$ ...(3)

Differentiating (3) w.r.t. x, we get $\qquad by_3 = -(0 + yy_3 + y_1y_2 + 2y_1y_2)$

$\Rightarrow$ $\qquad by_3 = -(yy_3 + 3y_1y_2)$ ...(4)

Elimination of *a*, *b* and *c*. Dividing (3) by (4), we get $\dfrac{by_2}{by_3} = \dfrac{-(1+yy_2+y_1^2)}{-(yy_3+3y_1y_2)}$

$$\Rightarrow \qquad \frac{y_2}{y_3} = \frac{1+yy_2+y_1^2}{yy_3+3y_1y_2} \qquad \Rightarrow \qquad yy_2y_3 + 3y_1y_2^2 = y_3 + yy_2y_3 + y_1^2y_3$$

$$\Rightarrow \qquad 3y_1y_2^2 = y_3 + y_1^2y_3 \qquad \Rightarrow \qquad (1 + y_1^2)y_3 - 3y_1y_2^2 = 0$$

NOTES

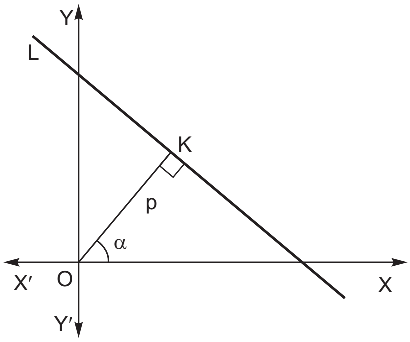
$$\therefore \quad \left[1+\left(\frac{dy}{dx}\right)^2\right]\frac{d^3y}{dx^3} - 3\,\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2 = 0.$$

This is the required differential equation.

Remark The differential equation obtained for each system in the above example is of order 'three'. This is so, because each system contained three arbitrary constants.

Example 5. *Form the differential equation of all lines in a plane which are at a constant distance p from the origin.*

Solution. The distance of the lines of the family from the origin is p. Let L be a line of this family. Draw OK perpendicular to this line. Let OK make angle α with the x-axis.



$\therefore$ The equation of the line L is

$$x\cos\alpha + y\sin\alpha = p.$$

$$\Rightarrow \quad x\cos\alpha + y\sin\alpha - p = 0 \qquad ...(1)$$

Differentiating (1) w.r.t. x, we get

$$\cos\alpha + y_1\sin\alpha - 0 = 0 \qquad ...(2)$$

Solving (1) and (2), we get

$$\frac{\cos\alpha}{0+py_1} = \frac{\sin\alpha}{-p-0} = \frac{1}{xy_1-y}$$

$$\therefore \quad \cos\alpha = \frac{py_1}{xy_1-y} \quad \text{and} \quad \sin\alpha = \frac{-p}{xy_1-y}$$

We have $\cos^2\alpha + \sin^2\alpha = 1$.

$$\therefore \quad \left(\frac{py_1}{xy_1-y}\right)^2 + \left(\frac{-p}{xy_1-y}\right)^2 = 1$$

$$\Rightarrow \quad p^2y_1^2 + p^2 = (xy_1-y)^2$$

$$\Rightarrow \quad p^2y_1^2 + p^2 = x^2y_1^2 + y^2 - 2xyy_1$$

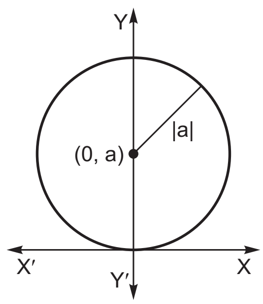
$$\Rightarrow \quad (p^2-x^2)y_1^2 + 2xyy_1 + p^2 - y^2 = 0$$

$$\Rightarrow \quad (p^2-x^2)\left(\frac{dy}{dx}\right)^2 + 2xy\frac{dy}{dx} + p^2 - y^2 = 0.$$

This is the required differential equation.

Example 6. *Form the differential equation of the system of circles touching the x-axis at the origin.*

Solution. The circles in the system will have their centres on the y-axis. Let $(0, a)$ be the centre of a circle touching the x-axis at the origin.



$\therefore$ The radius this circle must be $|a|$*, otherwise the circle will not touch the x-axis.

$\therefore$ The equation of the system of circle is

$$(x-0)^2 + (y-a)^2 = (|a|)^2$$

or $\quad x^2 + y^2 - 2ay = 0,\quad$ where a is an arbitrary constant.

***Why this step** If the centre of the circle is below the origin, then 'a' is negative. For such a circle, the radius of circle is $-a$, which is equal to $|a|$.

Differentiating w.r.t. x, we get

$$2x + 2yy_1 - 2ay_1 = 0 \quad \ldots(1)$$

Elimination of *a*.

(1) $\Rightarrow$ $$a = \frac{x + yy_1}{y_1}$$

NOTES

Putting the value of a in $x^2 + y^2 - 2ay = 0$, we get

$$x^2 + y^2 - 2\left(\frac{x + yy_1}{y_1}\right)y = 0.$$

$\Rightarrow$ $x^2y_1 + y^2y_1 - 2xy - 2y^2y_1 = 0 \Rightarrow (x^2 - y^2)y_1 = 2xy$

$\therefore$ $\boldsymbol{(x^2 - y^2)\dfrac{dy}{dx} = 2xy}$. This is the required differential equation.

> **Remark.** This differential equation also represent the system of circles passing through the origin and having centre on the y-axis.

Example 7. (*i*) *Form the differential equation of all parabolas with latus rectum '4a' and whose axes are parallel to the x-axis.*

(*ii*) *Form the differential equation of all parabolas whose axes are parallel to the y-axis.*

Solution. (*i*) The equation of a parabola with latus rectum '$4a$' and axis parallel to the x-axis is

$$(y - k)^2 = 4a(x - h), \quad \ldots(1)$$

where h and k are arbitrary constants.

Differentiating (1) w.r.t. x, we get

$$2(y - k)(y' - 0) = 4a(1 - 0) \quad i.e., \quad (y - k)y' = 2a \quad \ldots(2)$$

Differentiating (2) w.r.t. x, we get

$$(y - k)y'' + (y' - 0)y' = 0 \quad i.e., \quad y - k = \frac{y'^2}{y''} \quad \ldots(3)$$

Elimination of *h* and *k*.

(2) and (3) $\Rightarrow$ $\dfrac{y'^2}{y''} \cdot y' = 2a \quad \Rightarrow \quad \boldsymbol{2a\dfrac{d^2y}{dx^2} - \left(\dfrac{dy}{dx}\right)^3 = 0}.$

This is the required differential equation.

(*ii*) The equation of a parabola whose axis is parallel to the y-axis is given by

$$(x - h)^2 = 4a(y - k), \quad \ldots(1)$$

where h, k and a are arbitrary constants.

(1) $\Rightarrow$ $2(x - h)(1 - 0) = 4a(y' - 0) \Rightarrow x - h = 2ay'$

$\Rightarrow$ $1 - 0 = 2ay'' \Rightarrow 0 = 2ay''' \Rightarrow y''' = 0$

$(\because \; a \neq 0)$

$\Rightarrow$ $\boldsymbol{\dfrac{d^3y}{dx^3} = 0}$. This is the required differential equation.

NOTES

Example 8. *Form the differential equation representing the family of ellipses having foci on the x-axis and centre at the origin.*

Solution. Let the equation of the family of ellipses be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \ldots(1)$$

where a and b are parameters and $a > b > 0$.

Differentiating (1) w.r.t x, we get

$$\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0 \quad \Rightarrow \quad \frac{yy'}{x} = -\frac{b^2}{a^2} \quad \ldots(2)$$

Differentiating (2) w.r.t. x, we get

$$\frac{x[yy'' + y'y'] - yy' \cdot 1}{x^2} = 0 \quad i.e., \quad xyy'' + xy'^2 - yy' = 0.$$

$$\Rightarrow \quad \boldsymbol{xy \frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0}.$$

This is the required differential equation.

EXERCISE B

1. Form the differential equation of the family of curves given by:
   (*i*) $y = kx + k^2 + k^3$ (*ii*) $y + \lambda \sin x = 0$.
2. Form the differential equation of all straight lines passing through the origin.
3. Form the differential equation of the family of all non-vertical lines $y = mx + c$, in the xy-plane.
4. (*i*) Form a differential equation of the family of curves $y = a \sin (bx + c)$ where a and c being arbitrary constants.
   (*ii*) Form a differential equation of the family of curves $y = a \sin (bx + c)$ where a, b and c being arbitrary constants.
5. Obtain a differential equation that should be satisfied by the family of concentric circles $x^2 + y^2 = a^2$.
6. Form a differential equation of the family of circles given by $x^2 + y^2 = 2ax$.
7. Form the differential equation of the family of curves given by:
   (*i*) $y = \text{A}e^{2x} + \text{B}e^{-2x}$ (*ii*) $y = ax + bx^2$
   (*iii*) $xy = \text{C} \cos x$ (*iv*) $y = \frac{\text{A}}{r} + \text{B}$.
8. Form the differential equation of the family of curves given by:
   (*i*) $y = e^x(a \cos x + b \sin x)$, where a and b are arbitrary constants.
   (*ii*) $xy = \text{A}e^x + \text{B}e^{-x} + x^2$, where A and B are arbitrary constants.
   (*iii*) $y^2 = a(b - x^2)$, where a and b are arbitrary constants.
   (*iv*) $y = e^{2x}(a + bx)$, where a and b are arbitrary constants.
9. (*i*) Form the differential equation of the system of circles touching the y-axis at the origin.
   (*ii*) Form the differential equation of the system of circles which passes through the origin and having centres on the x-axis.
10. (*i*) Form the differential equation of all circles in the first quadrant which touch the coordinate axes.

(*ii*) Form the differential equation of all circles in the second quadrant and touching the coordinate axes.

11. (*i*) Form the differential equation of the family of circles of radius 2 units and having centre on the x-axis.

(*ii*) Form the differential equation of the family of circles having centre on the y-axis and radius 3 units.

12. (*i*) Form the differential equation of the family of circles $(x-a)^2 + (y-b)^2 = r^2$ by eliminating a and b.

(*ii*) Form the differential equation of the family of circles having radii 3.

13. Form the differential equation of all circles in the xy-plane.

14. (*i*) Form the differential equation of the family of parabolas having vertex at the origin and axis along the positive y-axis.

(*ii*) Form the differential equation of the family of parabolas having vertex at the origin and axis along the positive x-axis.

15. Form the differential equation of the family of ellipses having foci on the y-axis and centre at the origin.

16. Form the differential equation of the family of hyperbolas having foci on the x-axis and centre at the origin.

17. Show that the differential equation of which $x^2 - y^2 = c(x^2 + y^2)^2$ is a solution is

$$(x^3 - 3xy^2)dx = (y^3 - 3x^2y)\,dy.$$

Answers

1. (*i*) $y = x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^3$ (*ii*) $\frac{dy}{dx} = y \cot x$

2. $y = x\frac{dy}{dx}$ **3.** $\frac{d^2y}{dx^2} = 0$

4. (*i*) $\frac{d^2y}{dx^2} + b^2y = 0$ (*ii*) $y\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2}\frac{dy}{dx} = 0$

5. $x + y\frac{dy}{dx} = 0$ **6.** $2xy\frac{dy}{dx} + x^2 - y^2 = 0$

7. (*i*) $\frac{d^2y}{dx^2} - 4y = 0$ (*ii*) $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$

(*iii*) $x\frac{dy}{dx} + y + xy\tan x = 0$ (*iv*) $r\frac{d^2y}{dr^2} + 2\frac{dy}{dr} = 0$

8. (*i*) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$ (*ii*) $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = xy - x^2 + 2$

(*iii*) $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$ (*iv*) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$

9. (*i*) $2xy\frac{dy}{dx} + x^2 - y^2 = 0$ (*ii*) $2xy\frac{dy}{dx} + x^2 - y^2 = 0$

10. (*i*) $(x-y)^2\left(1 + \left(\frac{dy}{dx}\right)^2\right) = \left(x + y\frac{dy}{dx}\right)^2$ (*ii*) $(x+y)^2\left[1 + \left(\frac{dy}{dx}\right)^2\right] = \left(x + y\frac{dy}{dx}\right)^2$

11. (*i*) $y^2\left(1 + \left(\frac{dy}{dx}\right)^2\right) = 4$ (*ii*) $(x^2 - 9)\left(\frac{dy}{dx}\right)^2 + x^2 = 0$

12. (*i*) $r^2\left(\frac{d^2y}{dx^2}\right)^2 - \left(1 + \left(\frac{dy}{dx}\right)^2\right)^3 = 0$ (*ii*) $9\left(\frac{d^2y}{dx^2}\right)^2 - \left(1 + \left(\frac{dy}{dx}\right)^2\right)^3 = 0$

13. $\left(1+\left(\frac{dy}{dx}\right)^2\right)\frac{d^3y}{dx^3} - 3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2 = 0$

14. (*i*) $x\frac{dy}{dx} - 2y = 0$ (*ii*) $y^2 - 2xy\frac{dy}{dx} = 0$

15. $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$ 16. $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0.$

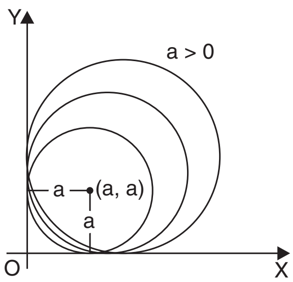
Hints

2. $y = mx \Rightarrow \frac{dy}{dx} = m \Rightarrow y = x\frac{dy}{dx}.$

4. (*i*) $y = a\sin(bx+c) \Rightarrow y_1 = ab\cos(bx+c) \Rightarrow y_2 = -ab^2\sin(bx+c) = -b^2y$
$\Rightarrow y_2 + b^2y = 0.$
(*ii*) $y = a\sin(bx+c) \Rightarrow y_1 = ab\cos(bx+c) \Rightarrow y_2 = -ab^2\sin(bx+c)$
$\Rightarrow y_3 = -ab^3\cos(bx+c)$
$\therefore\ y_2 = -b^2y$ and $y_3 = -b^2y_1 \Rightarrow \frac{y_2}{y} = \frac{y_3}{y_1} \Rightarrow yy_3 = y_1y_2.$

8. (*i*) $y = e^x(a\cos x + b\sin x)$
$\Rightarrow y_1 = e^x(a\cos x + b\sin x) + e^x(-a\sin x + b\cos x)$
$\Rightarrow y_1 = y + e^x(-a\sin x + b\cos x)$
$\Rightarrow y_2 = y_1 + e^x(-a\sin x + b\cos x) + e^x(-a\cos x - b\sin x)$
$\Rightarrow y_2 = y_1 + (y_1 - y) + (-y) \Rightarrow y_2 - 2y_1 + 2y = 0.$
(*ii*) $xy = Ae^x + Be^{-x} + x^2 \Rightarrow xy_1 + y = Ae^x - Be^{-x} + 2x$
$\Rightarrow (xy_2 + y_1) + y_1 = (Ae^x + Be^{-x}) + 2 \Rightarrow xy_2 + 2y_1 = (xy - x^2) + 2.$
(*iii*) $y^2 = a(b - x^2) \Rightarrow 2yy_1 = a(-2x) \Rightarrow yy_1 = -ax$
$\Rightarrow yy_2 + y_1 . y_1 = -a \quad \therefore\ yy_2 + y_1^2 = \frac{yy_1}{x}.$

10. (*i*) Let the equation of the circle be
$(x-a)^2 + (y-a)^2 = a^2$, where a is an arbitrary constant.
$\therefore\ 2(x-a) + 2(y-a)y_1 = 0$
$\Rightarrow x - a + yy_1 - ay_1 = 0 \Rightarrow a = \frac{x + yy_1}{1 + y_1}$



Putting the value of 'a' in the equation of circle, we get

$$\left(x - \frac{x+yy_1}{1+y_1}\right)^2 + \left(y - \frac{x+yy_1}{1+y_1}\right)^2 = \left(\frac{x+yy_1}{1+y_1}\right)^2$$

$\Rightarrow (x + xy_1 - x - yy_1)^2 + (y + yy_1 - x - yy_1)^2 = (x + yy_1)^2$
$\Rightarrow y_1^2(x-y)^2 + (y-x)^2 = (x+yy_1)^2.$

11. (*ii*) Let the equation of family of circles be
$(x-0)^2 + (y-a)^2 = 9.$ ...(1)
$\Rightarrow 2x + 2(y-a)y' = 0 \Rightarrow y - a = -\frac{x}{y'} \quad \therefore\ (1) \Rightarrow x^2 + \frac{x^2}{y'^2} = 9.$

12. (*i*) We have $(x-a)^2 + (y-b)^2 = r^2.$ ... (1)
(1) $\Rightarrow 2(x-a) + 2(y-b)y_1 = 0$ ... (2)
(2) $\Rightarrow 2 + 2(y-b)y_2 + 2(y_1)y_1 = 0$ ... (3)
(3) $\Rightarrow y - b = -\frac{1+y_1^2}{y_2}$ and
(2) $\Rightarrow x - a = -(y-b)y_1 = \left(\frac{1+y_1^2}{y_2}\right)y_1 = \frac{y_1 + y_1^3}{y_2}$
Now put the values of $x - a$ and $y - b$ in (1).

NOTES

13. The equation of a circle in xy-plane is

$x^2 + y^2 + 2gx + 2fy + c = 0$, where g, f, c are arbitrary constants. ... (1)

(1) $\Rightarrow$ $2x + 2yy_1 + 2g + 2fy_1 + 0 = 0$ $\Rightarrow$ $x + yy_1 + g + fy_1 = 0$ ... (2)

(2) $\Rightarrow$ $1 + yy_2 + y_1^2 + 0 + fy_2 = 0$ $\Rightarrow$ $(y + f)y_2 + {y_1}^2 + 1 = 0$ ...(3)

(3) $\Rightarrow (y + f)y_3 + y_1y_2 + 2y_1y_2 + 0 = 0$ $\Rightarrow$ $(y + f)y_3 + 3y_1y_2 = 0$ ...(4)

Multiply (3) by y_3, (4) by y_2 and subtract.

14. (i) Take $x^2 = 4ay$ as the equation of the family of parabolas.

15. Take $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$, $a > b > 0$ as the equation of the family of ellipse.

16. Take $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ as the equation of the family of hyperbolas.

17. We have $x^2 - y^2 = c(x^2 + y^2)^2$. ...(1)

(1) $\Rightarrow$ $2x - 2yy_1 = c.2(x^2 + y^2)(2x + 2yy_1)$

$\Rightarrow$ $x - yy_1 = 2c(x^2 + y^2)(x + yy_1)$ ...(2)

Divide (2) by (1) and simplify.

SOLUTION OF A DIFFERENTIAL EQUATION

A **solution of a differential equation** is a functional relation between the variables involved which satisfies the given differential equation.

A solution of a differential equation is called the **general solution** (or **complete solution**), if it contains as many arbitrary constants as the order of the differential equation.

Illustration $y = Cx^4$ is the *general solution* of the differential equation $x\dfrac{dy}{dx} - 4y = 0$, because the general solution contains one arbitrary constant 'C' and the order of the differential equation $x\dfrac{dy}{dx} - 4y = 0$ is also 'one'.

A solution obtained by giving particular values to arbitrary constants in the general solution of a differential equation is called a **particular solution** of the differential equation, under consideration.

Illustration $y = 7x^4$ is a *particular solution* of the differential equation $x\dfrac{dy}{dx} - 4y = 0$, because this solution has been obtained by giving a particular value '7' to the arbitrary constant 'C' in the general solution.

SOLVED EXAMPLES

Example 9. *Show that $y = be^x + ce^{2x}$ is a solution of $y_2 - 3y_1 + 2y = 0$.*

Solution. We have $y = be^x + ce^{2x}$.

$\therefore$ $y_1 = be^x + 2ce^{2x}$ and $y_2 = be^x + 4ce^{2x}$

$\therefore$ $y_2 - 3y_1 + 2y = (be^x + 4ce^{2x}) - 3(be^x + 2ce^{2x}) + 2(be^x + ce^{2x})$

$= be^x(1 - 3 + 2) + ce^{2x}(4 - 6 + 2) = be^x(0) + ce^{2x}(0) = 0.$

$\therefore$ $y = be^x + ce^{2x}$ is a solution of $y_2 - 3y_1 + 2y = 0$.

NOTES

Example 10. *Show that* $y = Ax + \frac{B}{x}$ *is a solution of* $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0.$

Solution. We have $y = Ax + \frac{B}{x}$.

$$\therefore \quad \frac{dy}{dx} = A + B(-1)x^{-2} = A - \frac{B}{x^2}$$

and
$$\frac{d^2y}{dx^2} = 0 + B(-1)(-2)\,x^{-3} = \frac{2B}{x^3}$$

$$\therefore \quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2\left[\frac{2B}{x^3}\right] + x\left[A - \frac{B}{x^2}\right] - \left[Ax + \frac{B}{x}\right]$$

$$= \frac{2B}{x} + xA - \frac{B}{x} - Ax - \frac{B}{x} = 0$$

$$\therefore \quad y = Ax + \frac{B}{x} \text{ is a solution of } x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0.$$

Example 11. *Show that* $y = \sqrt{A^2 - x^2}$, $x \in (-A, A)$ *is a solution of* $x + y\frac{dy}{dx} = 0,\ y \neq 0.$

Solution. We have $y = \sqrt{A^2 - x^2}$. $\therefore \frac{dy}{dx} = \frac{1}{2}(A^2 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{A^2 - x^2}}$

$$\therefore \quad x + y\frac{dy}{dx} = x + (\sqrt{A^2 - x^2})\left(\frac{-x}{\sqrt{A^2 - x^2}}\right) = x + (-x) = 0.$$

$$\therefore \quad y = \sqrt{A^2 - x^2} \text{ is a solution of } x + y\frac{dy}{dx} = 0.$$

EXERCISE C

1. Show that $x^2 + 4y = 0$ is a solution of $\left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx} - y = 0.$
2. Show that $y = \sqrt{1 + x^2}$ is a solution of $y' = \frac{xy}{1 + x^2}$.
3. Show that $y = \frac{1}{2x} + Ax + B$ is a solution of $x^3\frac{d^2y}{dx^2} = 1.$
4. Show that $y = a\cos x + b\sin x$ is a solution of $y'' + y = 0.$
5. Show that $y = 3\cos(\log x) + 4\sin(\log x)$ is a solution of $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$
6. Show that $y = ae^{2x} + be^{-x}$ is a solution of $y_2 - y_1 - 2y = 0.$
7. Show that $y = e^{3x}(A + Bx)$ is a solution of $y_2 - 6y_1 + 9y = 0.$
8. Show that $y = c_1e^{ax}\cos bx + c_2e^{ax}\sin bx$ is a solution of $y_2 - 2ay_1 + (a^2 + b^2)y = 0.$
9. Show that $y = \cos(\cos x)$ is a solution of $\frac{d^2y}{dx^2} - \cot x\frac{dy}{dx} + y\sin^2 x = 0.$
10. Show that $x + y = \tan^{-1} y$ is a solution of $y^2y' + y^2 + 1 = 0.$
11. Show that $y = x\sin x$ is a solution of $xy' = y + x\sqrt{x^2 - y^2}$ $(x \neq 0$ and $x > y$ or $x < -y)$.

12. Show that $y - \cos y = x$ is a solution of $(y \sin y + \cos y + x)y' = y$.

13. Show that $x^2 = 2y^2 \log y$ is a solution of $(x^2 + y^2)\dfrac{dy}{dx} - xy = 0$.

14. Show that $y = c_1e^x + c_2e^{-x}$ is the general solution of $\dfrac{d^2y}{dx^2} - y = 0$.

15. Show that $y = e^x + 1$ is a solution of $y'' - y' = 0$.

16. Show that $y = x^2 + 2x + C$ is a solution of $y' - 2x - 2 = 0$.

17. Show that $y = \cos x + C$ is a solution of $y' + \sin x = 0$.

18. Show that $y = Ax$ is a solution of $xy' = y$, $x \neq 0$.

19. Show that $y = ae^x + be^{-x} + x^2$ is a solution of $\dfrac{d^2y}{dx^2} - y + x^2 - 2 = 0$.

20. Show that $y = e^x (a \cos x + b \sin x)$ is solution of $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 0$.

21. Show that $x^2 - y^2 = c(x^2 + y^2)^2$ is a solution of $(x^3 - 3xy^2)dx = (y^3 - 3x^2y)dy$.

22. Can $y = ax + \dfrac{b}{a}$ be a solution of the following differential equation

$$y = x\frac{dy}{dx} + \frac{b}{\dfrac{dy}{dx}}?$$

If no, find the solution of the given differential equation. (*CBSE 2018 SP*)

Answer

22. Yes.

Hints

3. $y = \dfrac{1}{2x} + Ax + B \Rightarrow y_1 = -\dfrac{1}{2x^2} + A \Rightarrow y_2 = -\dfrac{1}{2}(-2)\,x^{-3} \Rightarrow x^3y_2 = 1.$

5. $y = 3\cos(\log x) + 4\sin(\log x) \Rightarrow y_1 = \dfrac{-3\sin(\log x)}{x} + \dfrac{4\cos(\log x)}{x}$

$\Rightarrow xy_1 = -3\sin(\log x) + 4\cos(\log x)$

$\Rightarrow xy_2 + 1 \cdot y_1 = \dfrac{-3\cos(\log x)}{x} - \dfrac{4\sin(\log x)}{x} = -\dfrac{y}{x}.$

9. $y = \cos(\cos x) \Rightarrow y_1 = -\sin(\cos x) \cdot -\sin x = \sin x \cdot \sin(\cos x)$

$\Rightarrow y_2 = \cos x \cdot \sin(\cos x) + \sin x \cdot \cos(\cos x) \cdot -\sin x$

$\Rightarrow y_2 = (\cos x)\left(\dfrac{y_1}{\sin x}\right) - (\sin^2 x)\,y.$

10. $x + y = \tan^{-1} y \Rightarrow 1 + y' = \dfrac{1}{1+y^2} \cdot y' \Rightarrow (1 + y^2)(1 + y') = y'$

$\Rightarrow 1 + y^2 + y^2y' = 0.$

11. $y = x \sin x \Rightarrow y' = \sin x + x \cos x$

$\Rightarrow xy' = x\sin x + x^2\cos x = y + x^2\sqrt{1-\sin^2 x} = y + x^2\sqrt{1-\dfrac{y^2}{x^2}} = y + x\sqrt{x^2 - y^2}.$

12. $y - \cos y = x \Rightarrow y' + (\sin y)y' = 1$ ...(1)

Also, $(y \sin y + \cos y + x)\,y' = (y \sin y + y - x + x)\,y'$

$= (y \sin y + y)\,y' = y(\sin y + 1)\,y' = y(1) = y.$ (By using (1))

13. $x^2 = 2y^2 \log y \Rightarrow 2x = 4yy' \log y + \dfrac{2y^2}{y}\,y' \Rightarrow x = (2y \log y + y)\,y'$

$\Rightarrow xy = (2y^2 \log y + y^2)\,y' \Rightarrow xy = (x^2 + y^2)\,y'.$

NOTES

INITIAL VALUE PROBLEM

A differential equation with given initial conditions is called an **initial value problem**.

$\frac{dy}{dx} = y\sec x,\ y(0) = 1$ is an initial value problem, because the solution of the differential equation $\frac{dy}{dx} = y\sec x$ is also to satisfy the initial condition $y(0) = 1$.

SOLVED EXAMPLES

Example 12. *Show that* $y = 2 - \frac{3x}{2x+1}$ *is a solution of the initial value problem:*

$$y - x\frac{dy}{dx} = 2\left(1 + x^2\frac{dy}{dx}\right),\ y(1) = 1.$$

Solution. We have $y = 2 - \frac{3x}{2x+1}$. ...(1)

$x = 1 \Rightarrow \quad y = 2 - \frac{3(1)}{2(1)+1} = 2 - 1 = 1 \quad \therefore \quad y(1) = 1$

(1) $\Rightarrow \quad \frac{dy}{dx} = 0 - \frac{(2x+1)3 - 3x(2)}{(2x+1)^2} = -\frac{3}{(2x+1)^2}$

$\therefore \quad y - x\frac{dy}{dx} = 2 - \frac{3x}{2x+1} - x\left(-\frac{3}{(2x+1)^2}\right) = 2 - \frac{3x}{2x+1} + \frac{3x}{(2x+1)^2}$

$$= \frac{2(2x+1)^2 - 3x(2x+1) + 3x}{(2x+1)^2} = \frac{2x^2+8x+2}{(2x+1)^2}$$

and $\quad 2\left(1 + x^2\frac{dy}{dx}\right) = 2\left(1 + x^2\left(-\frac{3}{(2x+1)^2}\right)\right)$

$$= 2\left(\frac{(2x+1)^2 - 3x^2}{(2x+1)^2}\right) = \frac{2x^2+8x+2}{(2x+1)^2}$$

$\therefore \quad y - x\frac{dy}{dx} = 2\left(1 + x^2\frac{dy}{dx}\right)$

$\therefore \quad y = 2 - \frac{3x}{2x+1}$

is a solution of the given initial value problem.

Example 13. *Show that* $y = x\sin 3x$ *is a solution of the initial value problem:*

$$\frac{d^2y}{dx^2} + 9y - 6\cos 3x = 0,\ y(0) = 0.$$

Solution. We have $y = x\sin 3x$. ...(1)

$x = 0 \Rightarrow \quad y = 0 \cdot \sin 3(0) = 0 \quad \therefore \quad y(0) = 0$

(1) $\Rightarrow \quad \frac{dy}{dx} = x \cdot 3\cos 3x + 1 \cdot \sin 3x = 3x\cos 3x + \sin 3x$

NOTES

$$\Rightarrow \qquad \frac{d^2y}{dx^2} = (3x\,(-3 \sin 3x) + 3.1.\cos 3x) + 3 \cos 3x$$
$$= -9x \sin 3x + 6 \cos 3x$$

$$\therefore \quad \frac{d^2y}{dx^2} + 9y - 6 \cos 3x = -9x \sin 3x + 6 \cos 3x + 9(x \sin 3x) - 6 \cos 3x = 0$$

$\therefore$ $y = x \sin 3x$ is a solution of the given initial value problem.

EXERCISE D

1. Show that $y = e^x$ is a solution of the initial value problem:
$$\frac{dy}{dx} = y,\ y(0) = 1.$$
2. Show that $y = \sin x + \cos x$ is a solution of the initial value problem:
$$\frac{d^2y}{dx^2} + y = 0,\ y(0) = 1,\ y'(0) = 1.$$
3. Show that $y = xe^x + e^x$ is a solution of the initial value problem:
$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0,\ y(0) = 1,\ y'(0) = 2.$$
4. Show that $3x^2y = 2x + y$ is a solution of the initial value problem:
$x^2\,dy + (xy + y^2)\,dx = 0,\ y(1) = 1.$
5. Show that $2y = x(x + y)$ is a solution of the initial value problem:
$$x^2 \frac{dy}{dx} = y^2 + 2xy,\ y(1) = 1.$$

SOLUTION OF A DIFFERENTIAL EQUATION BY THE METHOD OF SEPARATION OF VARIABLES

Let us consider the differential equation $\frac{dy}{dx} = f(x)\,g(y)$, ...(1)

where $f(x)$ and $g(y)$ are some functions of x and y respectively.

We know that

$$dy = \left(\frac{dy}{dx}\right) dx,$$

where dx and dy are respectively the differentials of the variables x and y.

$\therefore$ (1) $\Rightarrow$ $dy = f(x)\,g(y)\,dx$

$\Rightarrow$ $\frac{dy}{g(y)} = f(x)\,dx,$ provided $g(y) \neq 0$ ...(2)

In equation (2), the expressions involving y are on one side and the expressions involving x are on the other side.

Such a differential equation is said to be with *variables separable*.

Integrating equation (3), we get

$\int \frac{dy}{g(y)} = \int f(x)\,dx + C$, **where C is an arbitrary constant**.

This represents the general solution of the differential equation (1).

NOTES

Working Steps for Solving $\frac{dy}{dx} = f(x)\, g(y)$

Step I. Identify the functions f(x) and g(y).

Step II. Bring expressions involving x on one side and expressions involving y on the other side. Always keep dx and dy in the numerators.

Step III. Integrate both sides and add arbitrary constant 'C' only on one side. This gives the required general solution.

Step IV. If some initial condition is given, then find the value of the arbitrary constant 'C', so that the initial condition is satisfied. Put the value of 'C' in the general solution to get the required particular solution.

Tpye I. Solution of $\frac{dy}{dx} = f(x)$

If $g(y) = 1$, then $\frac{dy}{dx} = f(x)\, g(y)$ reduces to $\frac{dy}{dx} = f(x)$.

SOLVED EXAMPLES

Example 14. *Solve the differential equations:*

(*i*) $(e^x + e^{-x})dy - (e^x - e^{-x})dx = 0$ (*ii*) $\frac{dy}{dx} = \frac{1 - \cos x}{1 + \cos x}$.

Solution. (*i*) We have $(e^x + e^{-x})\, dy = (e^x - e^{-x})\, dx$.

$\Rightarrow$ $dy = \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx$ (Variables are separate)

Integrating, we get $\int 1 \cdot dy = \int \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx + C$.

$\Rightarrow$ $y = \log|e^x + e^{-x}| + C$ $\left(\because \int \frac{f'(x)}{f(x)}\, dx = \log|f(x)|\right)$

$\Rightarrow$ $\mathbf{y = \log(e^x + e^{-x}) + C}$.

(*ii*) We have $\frac{dy}{dx} = \frac{1 - \cos x}{1 + \cos x}$ *i.e.*, $dy = \frac{1 - \cos x}{1 + \cos x}\, dx$. (Variables are separate)

Integrating, we get $\int 1 \cdot dy = \int \frac{1 - \cos x}{1 + \cos x}\, dx + C$.

$\Rightarrow$ $y = \int \frac{2\sin^2 \frac{x}{2}}{2\cos^2 \frac{x}{2}}\, dx + C$

$\Rightarrow$ $y = \int \left(\sec^2 \frac{x}{2} - 1\right) dx + C$ $\Rightarrow$ $\mathbf{y = 2\tan \frac{x}{2} - x + C}$.

Example 15. *Solve the differential equations:*

(*i*) $(1+x^2)\dfrac{dy}{dx} - x = 2\tan^{-1} x$ (*ii*) $\cos x \dfrac{dy}{dx} - \cos 2x = \cos 3x.$

NOTES

Solution. (*i*) We have $(1+x^2)\dfrac{dy}{dx} - x = 2\tan^{-1} x.$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x + 2\tan^{-1} x}{1+x^2}$$

$$\Rightarrow \quad dy = \left(\frac{x}{1+x^2} + 2\,\frac{\tan^{-1} x}{1+x^2}\right) dx \qquad \text{(Variables are separate)}$$

$$\Rightarrow \quad dy = \left(\frac{1}{2}\cdot\frac{2x}{1+x^2} + 2\cdot\tan^{-1} x\cdot\frac{1}{1+x^2}\right) dx$$

Integrating, we get

$$\int dy = \frac{1}{2}\int \frac{2x}{1+x^2}\,dx + 2\int \tan^{-1} x\cdot\frac{1}{1+x^2}\,dx + \text{C}.$$

$$\Rightarrow \quad y = \frac{1}{2}\log\,|1+x^2| + 2\,\frac{(\tan^{-1} x)^2}{2} + \text{C}$$

$$\Rightarrow \quad \boldsymbol{y = \frac{1}{2}\log(1+x^2) + (\tan^{-1} x)^2 + \text{C}}.$$

(*ii*) We have $\cos x\,\dfrac{dy}{dx} - \cos 2x = \cos 3x.$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\cos 2x + \cos 3x}{\cos x}$$

$$\Rightarrow \quad dy = \frac{2\cos^2 x - 1 + 4\cos^3 x - 3\cos x}{\cos x}\,dx \qquad \text{(Variables are separate)}$$

$$\Rightarrow \quad dy = \left(2\cos x + 4\cos^2 x - 3 - \frac{1}{\cos x}\right)dx.$$

$$\Rightarrow \quad dy = \left(2\cos x + 4\left(\frac{1+\cos 2x}{2}\right) - 3 - \sec x\right)dx$$

$$\Rightarrow \quad dy = (2\cos x + 2 + 2\cos 2x - 3 - \sec x)dx$$

$$\Rightarrow \quad dy = (2\cos x + 2\cos 2x - 1 - \sec x)dx$$

Integrating, we get

$$y = 2\sin x + 2.\frac{\sin 2x}{2} - x - \log|\sec x + \tan x| + \text{C}.$$

$$\Rightarrow \quad \boldsymbol{y = 2\sin x + \sin 2x - x - \log|\sec x + \tan x| + \text{C}}.$$

Example 16. *Solve the differential equation:*

$$(x^3 + x^2 + x + 1)\,\frac{dy}{dx} = 2x^2 + x,\ y = 1 \text{ when } x = 0.$$

Solution. We have $(x^3 + x^2 + x + 1)\,\dfrac{dy}{dx} = 2x^2 + x.$

$$\Rightarrow \quad dy = \frac{2x^2 + x}{x^3 + x^2 + x + 1}\,dx \qquad \text{(Variables are separate)}$$

NOTES

Integrating, we get

$$\int dy = \int \frac{2x^2 + x}{x^3 + x^2 + x + 1}\, dx + C \quad \Rightarrow \quad y = \int \frac{2x^2 + x}{(x+1)(x^2+1)}\, dx + C$$

$$\Rightarrow \qquad y = \int \left(\frac{\frac{1}{2}}{x+1} + \frac{\frac{3}{2}x - \frac{1}{2}}{x^2+1} \right) dx + C$$

(By resolving into partial fractions)

$$\Rightarrow \qquad y = \frac{1}{2}\log|x+1| + \frac{3}{4}\log(x^2+1) - \frac{1}{2}\tan^{-1}x + C$$

Also, $y = 1$ when $x = 0$ ...(2)

$\therefore$ (1) $\Rightarrow$ $1 = \frac{1}{2}\log|1| + \frac{3}{4}\log(1) - \frac{1}{2}(0) + C \Rightarrow C = 1$

$\therefore$ The required solution is

$$\mathbf{y = \frac{1}{2}\log|x+1| + \frac{3}{4}\log(x^2+1) - \frac{1}{2}\tan^{-1}x + 1.}$$

EXERCISE E

Find the general solution of the following differential equations (Q. No. 1–3):

1. (*i*) $x^2 \frac{dy}{dx} = 2$ (*ii*) $\frac{dy}{dx} = \frac{x}{x^2+1}$

(*iii*) $\frac{dy}{dx} = x^2 + \sin 3x$ (*iv*) $\frac{dy}{dx} = \frac{1 - \cos 4x}{1 + \cos 4x}$

2. (*i*) $(x+2)\frac{dy}{dx} = x^2 + 4x - 9$ (*ii*) $\sqrt{1 - x^6}\, dy = x^2\, dx$

(*iii*) $\frac{dy}{dx} = \log(x+1)$ (*iv*) $\frac{dy}{dx} = \frac{1}{\sin^4 x + \cos^4 x}$

3. (*i*) $\frac{dy}{dx} = x^5 \tan^{-1} x^3$ (*ii*) $\frac{dy}{dx} = \sin^3 x \cos^2 x + x e^x$

(*iii*) $\frac{1}{x}\frac{dy}{dx} = \tan^{-1} x$ (*iv*) $\frac{dy}{dx} = \sin^{-1} x$

4. Solve the following initial value problems:

(*i*) $x\frac{dy}{dx} + 1 = 0,\ y(-1) = 0$ (*ii*) $e^{dy/dx} = x + 1,\ y(0) = 5$

(*iii*) $x(x^2 - 1)\frac{dy}{dx} = 1,\ y(2) = 0$ (*iv*) $\sin\left(\frac{dy}{dx}\right) = k,\ y(0) = 1.$

Answers

1. (*i*) $y + \frac{2}{x} = C$ (*ii*) $y = \frac{1}{2}\log(x^2+1) + C$

(*iii*) $y = \frac{x^3}{3} - \frac{\cos 3x}{3} + C$ (*iv*) $y = \frac{1}{2}\tan 2x - x + C$

2. (*i*) $y = \frac{x^2}{2} + 2x - 13\log|x+2| + C$ (*ii*) $y = \frac{1}{3}\sin^{-1} x^3 + C$

(*iii*) $y = (x + 1) \log (x + 1) - x + C$ (*iv*) $y = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\tan x - \cot x}{\sqrt{2}}\right) + C$

3. (*i*) $6y = (x^6 + 1) \tan^{-1} x^3 - x^3 + C$ (*ii*) $y = \frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x + (x - 1) e^x + C$

(*iii*) $y = \frac{1}{2}(x^2 + 1) \tan^{-1} x - \frac{1}{2} x + C$ (*iv*) $y = x \sin^{-1} x + \sqrt{1 - x^2} + C$

4. (*i*) $y + \log |x| = 0$ (*ii*) $y = (x + 1) \log (x + 1) - x + 5$

(*iii*) $y = \frac{1}{2} \log \frac{4(x^2 - 1)}{3x^2}$ (*iv*) $y = x \sin^{-1} k + 1$.

NOTES

TYPE II. Solution of $\frac{dy}{dx} = g(y)$

If $f(x) = 1$, then $\frac{dy}{dx} = f(x)\, g(y)$ reduces to $\frac{dy}{dx} = g(y)$

SOLVED EXAMPLES

Example 17. *Solve the differential equations:*

(*i*) $\frac{dy}{dx} + \frac{1 + y^2}{y} = 0$ (*ii*) $\frac{dy}{dx} = \frac{1}{\log y}$.

Solution. (*i*) We have $\frac{dy}{dx} + \frac{1 + y^2}{y} = 0$.

$\Rightarrow \quad \frac{dy}{dx} = -\frac{1 + y^2}{y} \quad \Rightarrow \quad \frac{y}{1 + y^2}\, dy = -\, dx$

(Variables are separate)

Integrating, we get $\int \frac{y}{1 + y^2}\, dy = -\int dx + C$.

$\Rightarrow \quad \frac{1}{2} \int \frac{2y}{1 + y^2}\, dy = -x + C \quad \Rightarrow \quad \boldsymbol{\frac{1}{2} \log (1 + y^2) + x = C}$.

(*ii*) We have $\frac{dy}{dx} = \frac{1}{\log y}$.

$\Rightarrow \quad \log y\, dy = dx$ (Variables are separate)

Integrating, we get $\int \log y\, dy = \int dx + C$.

$\Rightarrow \quad \int \log y \cdot 1\, dy = x + C \quad \Rightarrow \quad (\log y)\, y - \int \frac{1}{y} \cdot y\, dy = x + C$

$\Rightarrow \quad \boldsymbol{y \log y - y = x + C}$.

Example 18. *Solve the differential equations:*

(*i*) $\frac{dy}{dx} + \cos^2 y = 0$ (*ii*) $\frac{dy}{dx} + \frac{1 + \cos y}{1 - \cos y} = 0$.

Solution. (*i*) We have $\frac{dy}{dx} + \cos^2 y = 0$.

$\Rightarrow \quad \frac{dy}{dx} = -\cos^2 y \quad \Rightarrow \quad \sec^2 y\, dy = -\, dx$ (Variables are separate)

NOTES

Integrating, we get

$$\int \sec^2 y\, dy = -\int dx + C \quad \Rightarrow \quad \mathbf{\tan y = -x + C}.$$

(*ii*) We have $\dfrac{dy}{dx} + \dfrac{1+\cos y}{1-\cos y} = 0.$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{1+\cos y}{1-\cos y} \quad \Rightarrow \quad \frac{1-\cos y}{1+\cos y}\, dy = -\, dx$$

(Variables are separate)

Integrating, we get

$$\int \frac{1-\cos y}{1+\cos y}\, dy = -\int dx + C.$$

$$\Rightarrow \quad \int \frac{2\sin^2 \frac{y}{2}}{2\cos^2 \frac{y}{2}}\, dy = -x + C \quad \Rightarrow \quad \int \left(\sec^2 \frac{y}{2} - 1\right) dy = -x + C$$

$$\Rightarrow \quad 2\tan \frac{y}{2} - y = -x + C \quad \Rightarrow \quad \mathbf{x - y + 2\tan \frac{y}{2} = C}.$$

EXERCISE F

Find the general solution of the following differential equations (Q. No. 1–3):

1. (*i*) $\dfrac{dy}{dx} + y = 1,\ y \neq 1$ (*ii*) $\dfrac{dy}{dx} = \dfrac{1}{y^2 + \sin y}$

(*iii*) $\dfrac{dy}{dx} = \dfrac{1+y^2}{y^3}$ (*iv*) $\dfrac{dy}{dx} = \sqrt{4 - y^2}$

2. (*i*) $y\, dy = \dfrac{dx}{\tan^{-1} y}$ (*ii*) $(\sin y - \cos y)dy = (\sin y + \cos y)dx$

(*iii*) $(y^5 \tan^{-1} y^3)dy = dx$ (*iv*) $y \log y\, dy = dx$

3. (*i*) $\dfrac{dy}{dx} = \sec y$ (*ii*) $\dfrac{dy}{dx} = \sin^2 y$

(*iii*) $\dfrac{dy}{dx} = \dfrac{1-\cos 2y}{1+\cos 2y}$ (*iv*) $(2y^2 + y)\dfrac{dy}{dx} = y^3 + y^2 + y + 1$

4. Solve the following initial value problems:

(*i*) $\dfrac{dy}{dx} + 2y^2 = 0,\ y(1) = 1$ (*ii*) $\dfrac{dy}{dx} + \cos^2 y = 0,\ y(0) = \dfrac{\pi}{4}.$

Answers

1. (*i*) $\log |1 - y| + x + C = 0$ (*ii*) $x = \dfrac{y^3}{3} - \cos y + C$

(*iii*) $x = \dfrac{y^2}{2} - \dfrac{1}{2}\log (y^2 + 1) + C$ (*iv*) $y = 2\sin (x + C)$

2. (*i*) $x = \dfrac{1}{2}(y^2 + 1)\tan^{-1} y - \dfrac{y}{2} + C$ (*ii*) $x + \log |\sin y + \cos y| = C$

(*iii*) $6x = (y^6 + 1)\tan^{-1} y^3 - y^3 + C$ (*iv*) $4x = 2y^2 \log y - y^2 + C$

3. (*i*) $x = \sin y + C$ (*ii*) $x + \cot y = C$

(*iii*) $x + \cot y + y = C$

(*iv*) $x = \frac{1}{2} \log |y + 1| + \frac{3}{4} \log (y^2 + 1) - \frac{1}{2} \tan^{-1} y + C$

4. (*i*) $y = \frac{1}{2x - 1}$ (*ii*) $x + \tan y = 1$.

NOTES

TYPE III. Solution of $\frac{dy}{dx} = f(x)\, g(y)$

SOLVED EXAMPLES

Example 19. *Solve the differential equations:*

(*i*) $(y + xy)dx + (x - xy^2)dy = 0$

(*ii*) $(y^3 + 1)(e^x + xe^x)\, dx - xe^x y^2\, dy = 0$.

Solution. (*i*) We have $(y + xy)dx + (x - xy^2)dy = 0$.

$\Rightarrow \quad y(1 + x)dx + x(1 - y^2)dy = 0 \quad \Rightarrow \quad y(1 + x)dx = -x(1 - y^2)dy$

$$\Rightarrow \quad \frac{1+x}{x} dx = \frac{y^2 - 1}{y} dy \qquad \text{(Variables are separate)}$$

Integrating, we get $\int \frac{1+x}{x}\, dx = \int \frac{y^2 - 1}{y}\, dy + C$.

$$\Rightarrow \quad \int \left(\frac{1}{x} + 1\right) dx = \int \left(y - \frac{1}{y}\right) dy + C$$

$$\Rightarrow \quad \log |x| + x = \frac{y^2}{2} - \log |y| + C$$

$$\Rightarrow \quad \mathbf{\log |xy| + x = \frac{y^2}{2} + C}.$$

(*ii*) We have $(y^3 + 1)(e^x + xe^x)dx - xe^x y^2\, dy = 0$.

$\Rightarrow \quad (y^3 + 1)e^x (1 + x)dx = xe^x y^2\, dy$

$$\Rightarrow \quad \frac{1+x}{x}\, dx = \frac{y^2}{y^3 + 1}\, dy \qquad \text{(Variables are separate)}$$

Integrating, we get $\int \left(\frac{1}{x} + 1\right) dx = \frac{1}{3} \int \frac{3y^2}{y^3 + 1}\, dy + C$.

$$\Rightarrow \quad \mathbf{\log |x| + x = \frac{1}{3} \log |y^3 + 1| + C}.$$

Example 20. *Solve the differential equations:*

(*i*) $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$

(*ii*) $e^x \tan y\, dx + (1 - e^x) \sec^2 y\, dy = 0$.

Solution. (*i*) We have $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$.

$\Rightarrow \quad \sec^2 x \tan y\, dx = -\sec^2 y \tan x\, dy$

NOTES

$$\Rightarrow \quad \frac{\sec^2 x}{\tan x}\,dx = -\frac{\sec^2 y}{\tan y}\,dy \qquad \text{(Variables are separate)}$$

Integrating, we get $\int \frac{\sec^2 x}{\tan x}\,dx = -\int \frac{\sec^2 y}{\tan y}\,dy + C.$

$\Rightarrow \quad \log|\tan x| = -\log|\tan y| + C \quad \Rightarrow \quad \log|\tan x \tan y| = C$

$\Rightarrow \quad |\tan x \tan y| = e^C \quad \Rightarrow \quad \tan x \tan y = \pm e^C$

$\Rightarrow \quad \mathbf{\tan x \tan y = C_1}.$ (Taking $C_1 = \pm e^C$)

(*ii*) We have $e^x \tan y\, dx + (1 - e^x) \sec^2 y\, dy = 0.$

$\Rightarrow \quad e^x \tan y\, dx = (e^x - 1) \sec^2 y\, dy$

$$\Rightarrow \quad \frac{e^x}{e^x - 1}\,dx = \frac{\sec^2 y}{\tan y}\,dy \qquad \text{(Variables are separate)}$$

Integrating, we get $\int \frac{e^x}{e^x - 1}\,dx = \int \frac{\sec^2 y}{\tan y}\,dy + C.$

$\Rightarrow \quad \log|e^x - 1| = \log|\tan y| + C$

$$\Rightarrow \quad \log\frac{|e^x - 1|}{|\tan y|} = C \quad \Rightarrow \quad \left|\frac{e^x - 1}{\tan y}\right| = e^C \quad \Rightarrow \quad \frac{e^x - 1}{\tan y} = \pm e^C$$

$\Rightarrow \quad \mathbf{e^x - 1 = C_1 \tan y}.$ (Taking $C_1 = \pm e^C$)

Example 21. *Solve:*

(*i*) $\sqrt{1 + x^2 + y^2 + x^2y^2} + xy\frac{dy}{dx} = 0$ (*ii*) $\log\left(\frac{dy}{dx}\right) = 3x + 4y.$

Solution. (*i*) We have $\sqrt{1 + x^2 + y^2 + x^2y^2} + xy\frac{dy}{dx} = 0.$

$$\Rightarrow \quad \sqrt{(1 + x^2)(1 + y^2)} + xy\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{\sqrt{1 + x^2}}{x}\,dx + \frac{y}{\sqrt{1 + y^2}}\,dy = 0$$

(Variables are separate)

Integrating, we get $\int \frac{\sqrt{1 + x^2}}{x}\,dx + \int \frac{y}{\sqrt{1 + y^2}}\,dy = C.$ ...(1)

$$\sqrt{1 + x^2} = z \quad \Rightarrow \quad 1 + x^2 = z^2 \quad \Rightarrow \quad 2x\,dx = 2z\,dz$$

$$\therefore \quad \int \frac{\sqrt{1 + x^2}}{x}\,dx = \int \frac{\sqrt{1 + x^2}}{x^2}\,x\,dx = \int \frac{z}{z^2 - 1}\,.\,z\,dz$$

$$= \int \frac{z^2}{z^2 - 1}\,dz = \int \frac{(z^2 - 1) + 1}{z^2 - 1}\,dz$$

$$= \int \left[1 + \frac{1}{(z - 1)(z + 1)}\right] dz = z + \int \left[\frac{1}{(z - 1)(2)} + \frac{1}{(-2)(z + 1)}\right] dz$$

$$= z + \frac{1}{2}(\log|z - 1| - \log|z + 1|) = z + \frac{1}{2}\log\left|\frac{z - 1}{z + 1}\right|$$

$$= \sqrt{1+x^2} + \frac{1}{2}\log\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right| = \sqrt{1+x^2} + \frac{1}{2}\log\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}$$

Also, $\sqrt{1+y^2} = z \Rightarrow 1 + y^2 = z^2 \Rightarrow 2y\,dy = 2z\,dz$

NOTES

$$\therefore \int \frac{y}{\sqrt{1+y^2}}dy = \int \frac{z\,dz}{z} = \int 1\,dz = z = \sqrt{1+y^2}$$

$$\therefore \ (1) \Rightarrow \boldsymbol{\sqrt{1+x^2} + \frac{1}{2}\log\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1} + \sqrt{1+y^2} = C}.$$

(*ii*) We have $\log\left(\frac{dy}{dx}\right) = 3x + 4y$.

$$\Rightarrow \quad \frac{dy}{dx} = e^{3x+4y} \Rightarrow \frac{dy}{dx} = e^{3x}e^{4y}$$

$$\Rightarrow \quad e^{-4y}\,dy = e^{3x}\,dx \qquad \text{(Variables are separate)}$$

Integrating, we get $\int e^{-4y}\,dy = \int e^{3x}\,dx + C$.

$$\Rightarrow \quad \boldsymbol{\frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} + C}.$$

Example 22. *Show that the general solution of the differential equation* $\frac{dy}{dx} + \frac{y^2+y+1}{x^2+x+1} = 0$ *is given by* $(x + y + 1) = A(1 - x - y - 2xy)$*, where A is a parameter.*

Solution We have $\frac{dy}{dx} + \frac{y^2+y+1}{x^2+x+1} = 0$, *i.e.*, $\frac{dy}{y^2+y+1} + \frac{dx}{x^2+x+1} = 0$.

(Variables are separate)

Integrating, we get $\int \frac{dy}{y^2+y+1} + \int \frac{dx}{x^2+x+1} = C$. ...(1)

$$\Rightarrow \int \frac{dy}{\left(y+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = C$$

$$\Rightarrow \frac{1}{\sqrt{3}/2}\tan^{-1}\left(\frac{y+\frac{1}{2}}{\sqrt{3}/2}\right) + \frac{1}{\sqrt{3}/2}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right) = C$$

$$\Rightarrow \frac{2}{\sqrt{3}}\left(\tan^{-1}\left(\frac{2y+1}{\sqrt{3}}\right) + \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)\right) = C$$

$$\Rightarrow \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + \tan^{-1}\left(\frac{2y+1}{\sqrt{3}}\right) = \frac{\sqrt{3}}{2}C = C_1, \text{ say}$$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{2x+1}{\sqrt{3}} + \frac{2y+1}{\sqrt{3}}}{1 - \frac{2x+1}{\sqrt{3}}\cdot\frac{2y+1}{\sqrt{3}}}\right) = C_1 \Rightarrow \frac{\sqrt{3}\,(2x+1+2y+1)}{3-(2x+1)(2y+1)} = \tan C_1$$

NOTES

$$\Rightarrow \quad \frac{2\sqrt{3}\,(x+y+1)}{2-4xy-2x-2y} = \tan C_1 \Rightarrow \frac{\sqrt{3}\,(x+y+1)}{1-x-y-2xy} = \tan C_1$$

$$\Rightarrow \quad x+y+1 = \frac{\tan C_1}{\sqrt{3}}\,(1-x-y-2xy)$$

$$\Rightarrow \quad \mathbf{x+y+1 = A(1-x-y-2xy),}$$

$$\textbf{where } \mathbf{A = \frac{\tan C_1}{\sqrt{3}}} \textbf{ is a parameter}.$$

Example 23. *Solve the following differential equations:*

(*i*) $ye^{x/y}\,dx = (xe^{x/y} + y^2)dy,\ y \neq 0$ (*ii*) $\dfrac{dy}{dx} = \dfrac{2x(\log x + 1)}{\sin y + y\cos y}$.

Solution. (*i*) We have $ye^{x/y}\,dx = (xe^{x/y} + y^2)dy$ *i.e.*, $e^{x/y}\dfrac{dx}{dy} = \dfrac{x}{y}e^{x/y} + y.$ ...(1)

Put $z = \dfrac{x}{y}$ $\therefore$ $x = zy$ and $\dfrac{dx}{dy} = z + y\dfrac{dz}{dy}$

$$\therefore \quad (1) \Rightarrow e^z\left(z + y\frac{dz}{dy}\right) = ze^z + y \Rightarrow e^z y\frac{dz}{dy} = y \Rightarrow e^z\frac{dz}{dy} = 1$$

$\Rightarrow$ $e^z dz = dy$ (Variables are separate)

Integrating, we get $\displaystyle\int e^z\,dz = \int dy + C$.

$$\Rightarrow \quad e^z = y + C \Rightarrow \mathbf{e^{x/y} = y + C}.$$

(*ii*) We have $\dfrac{dy}{dx} = \dfrac{2x(\log x + 1)}{\sin y + y\cos y}$

$$\Rightarrow \quad (\sin y + y\cos y)\,dy = 2x\,(\log x + 1)\,dx$$

$$\Rightarrow \quad \int(\sin y + y\cos y)dy = \int 2x\log x\,dx + \int 2x\,dx + C$$

$$\Rightarrow \quad \int \sin y\,dy + \int \underset{\text{I}}{y}\,.\,\underset{\text{II}}{\cos y}\,dy = 2\int \underset{\text{I}}{\log x}\,.\,\underset{\text{II}}{x}\,dx + x^2 + C$$

$$\Rightarrow -\cos y + \left[y\sin y - \int 1\,.\,\sin y\,dy\right] = 2\left[(\log x)\frac{x^2}{2} - \int\frac{1}{x}\,.\,\frac{x^2}{2}\,dx\right] + x^2 + C$$

$$\Rightarrow \quad -\cos y + y\sin y + \cos y = x^2\log x - \frac{x^2}{2} + x^2 + C$$

$$\mathbf{y\sin y = x^2\log x + \frac{x^2}{2} + C}.$$

Example 24. *Solve the following initial value problems:*

(*i*) $y' = y\tan 2x,\ y(0) = 2$ (*ii*) $2xy' = 3y,\ y(1) = 4.$

Solution. (*i*) We have $y' = y\tan 2x$.

$$\therefore \quad \frac{dy}{dx} = y\tan 2x \Rightarrow \frac{dy}{y} = \tan 2x\,dx$$

$$\therefore \quad \int\frac{dy}{y} = \int\tan 2x\,dx + C \Rightarrow \log|y| = \frac{\log|\sec 2x|}{2} + C.$$

Now, $y(0) = 2$ implies $\log|2| = \dfrac{\log|\sec 0|}{2} + C$

i.e., $\log 2 = \dfrac{\log 1}{2} + C$ or $C = \log 2$

$\therefore$ The required particular solution is $\log|y| = \dfrac{\log|\sec 2x|}{2} + \log 2.$

$\Rightarrow \quad \log|y| = \log|\sec 2x|^{1/2} + \log 2$

$\Rightarrow \quad \log|y| = \log \dfrac{2}{\sqrt{\cos 2x}}$ (Assuming sec $2x > 0$)

NOTES

$\Rightarrow \quad \mathbf{y = \dfrac{2}{\sqrt{\cos 2x}}}.$ (Assuming $y > 0$)

(*ii*) We have $\quad 2xy' = 3y \quad$ *i.e.,* $\quad 2x\dfrac{dy}{dx} = 3y \quad$ or $\quad \dfrac{dy}{y} = \dfrac{3}{2}\dfrac{dx}{x}.$

$\Rightarrow \quad \displaystyle\int \frac{dy}{y} = \frac{3}{2}\int \frac{dx}{x} + C \quad \Rightarrow \quad \log|y| = \frac{3}{2}\log|x| + C.$

Now, $y(1) = 4$ implies $\quad \log|4| = \dfrac{3}{2}\log|1| + C \quad$ or $\quad C = \log 4$

$\therefore \quad \log|y| = \dfrac{3}{2}\log|x| + \log 4 = \log 4|x|^{3/2}$

$\Rightarrow \quad \mathbf{y = 4x^{3/2}}.$ (Assuming $x > 0, y > 0$)

This is the required particular solution.

Example 25. *Solve the following initial value problems:*

(*i*) $\cos y\, dy + \cos x \sin y\, dx = 0,\ y(\pi/2) = \pi/2$

(*ii*) $\dfrac{dy}{dx} = e^{-y}\cos x,\ y(0) = 0.$

Solution. (*i*) We have $\quad \cos y\, dy + \cos x \sin y\, dx = 0.$

$\Rightarrow \quad \dfrac{\cos y\, dy}{\sin y} + \cos x\, dx = 0 \quad \Rightarrow \quad \cot y\, dy + \cos x\, dx = 0$

$\therefore \quad \displaystyle\int \cot y\, dy + \int \cos x\, dx = C \quad \Rightarrow \quad \log|\sin y| + \sin x = C$

Now, $\quad y\left(\dfrac{\pi}{2}\right) = \dfrac{\pi}{2}$ implies $\log\left|\sin\dfrac{\pi}{2}\right| + \sin\dfrac{\pi}{2} = C$ *i.e.,* $0 + 1 = C$ or $C = 1$

$\therefore$ The required particular solution is $\mathbf{\log|\sin y| + \sin x = 1}$.

(*ii*) We have $\quad \dfrac{dy}{dx} = e^{-y}\cos x.$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{\cos x}{e^y} \quad \Rightarrow \quad e^y\, dy = \cos x\, dx$

$\Rightarrow \quad \displaystyle\int e^y\, dy = \int \cos x\, dx + C \quad \Rightarrow \quad e^y = \sin x + C.$

Now $y(0) = 0$ implies $\quad e^0 = \sin 0 + C \quad$ or $\quad 1 = 0 + C$ or $C = 1$

$\therefore$ The required particular solution is $\mathbf{e^y = \sin x + 1}$.

Example 26. *Solve the following initial value problems:*

(*i*) $\dfrac{dy}{dx} = 1 + x^2 + y^2 + x^2y^2$ *given that* $y = 1$ *when* $x = 0.$

(*ii*) $(x - y)(dx + dy) = dx - dy$ *given that* $y = -1$ *when* $x = 0.$

Solution. (*i*) We have $\quad \dfrac{dy}{dx} = 1 + x^2 + y^2 + x^2y^2.$

$\Rightarrow \quad \dfrac{dy}{dx} = (1 + x^2)(1 + y^2) \quad \Rightarrow \quad \dfrac{dy}{1 + y^2} = (1 + x^2)\, dx$

NOTES

Integrating, we get

$$\tan^{-1} y = x + \frac{x^3}{3} + C. \quad ...(1)$$

Also, $y = 1$ when $x = 0$

$\therefore$ (1) $\Rightarrow$ $\tan^{-1}(1) = 0 + 0 + C \quad \therefore \quad C = \pi/4$

$\therefore$ (1) $\Rightarrow$ $\mathbf{\tan^{-1} y = x + \frac{x^3}{3} + \frac{\pi}{4}}$.

(*ii*) We have $(x - y)(dx + dy) = dx - dy$.

$\Rightarrow$ $dx + dy = \dfrac{dx - dy}{x - y} \quad \Rightarrow \quad d(x + y) = \dfrac{d(x - y)}{x - y}$

Integrating, we get

$$\int d(x+y) = \int \frac{d(x-y)}{x-y} + C \quad \Rightarrow \quad x + y = \log|x - y| + C \quad ...(1)$$

Also, $y = -1$ when $x = 0$

$\therefore$ (1) $\Rightarrow$ $0 - 1 = \log|0 - (-1)| + C \Rightarrow -1 = 0 + C \Rightarrow C = -1$

$\therefore$ (1) $\Rightarrow$ $\mathbf{x + y = \log|x - y| - 1}$.

EXERCISE G

Find the general solution of the following differential equations (Q. No. 1–2):

1. (*i*) $\dfrac{dy}{dx} = \dfrac{4y}{x(y-2)}$ (*ii*) $\dfrac{dy}{dx} + \sqrt{\dfrac{1-y^2}{1-x^2}} = 0$

(*iii*) $\dfrac{dy}{dx} = (e^x + 1)y$ (*iv*) $\dfrac{dy}{dx} = x^3 e^{-2y}$

2. (*i*) $x^5 \dfrac{dy}{dx} = -y^5$ (*ii*) $\dfrac{dy}{dx} = \dfrac{x+1}{2-y}$

(*iii*) $\dfrac{dy}{dx} = \dfrac{1+y^2}{1+x^2}$ (*iv*) $y \log y \, dx - x \, dy = 0$.

Find the general solution of the following differential equations (Q. No. 3–8):

3. (*i*) $y(1-x^2)\dfrac{dy}{dx} = x(1+y^2)$ (*ii*) $(1 + x)y \, dx + (1 - y)x \, dy = 0$

(*iii*) $y - a\dfrac{dy}{dx} = ay^2 + x\dfrac{dy}{dx}$ (*iv*) $\text{cosec}\, x \log y \dfrac{dy}{dx} + x^2y^2 = 0$.

4. (*i*) $(1 + x)(1 + y^2)dx + (1 + y)(1 + x^2)dy = 0$

(*ii*) $(x^2 - yx^2) \, dy + (y^2 + x^2y^2)dx = 0$

(*iii*) $y - x\dfrac{dy}{dx} = a\left(y^2 + \dfrac{dy}{dx}\right)$ (*iv*) $x\sqrt{1-y^2}\, dx + y\sqrt{1-x^2}\, dy = 0$

5. (*i*) $\dfrac{dy}{dx} = x + y + xy + 1$ (*ii*) $(1 - e^x)\sec^2 y \, dy - 2e^x \tan y \, dx = 0$

(*iii*) $(1 + y^2)(1 + \log x)dx + x \, dy = 0$ (*iv*) $\cos x \cos y \dfrac{dy}{dx} + \sin x \sin y = 0$

6. (*i*) $(e^x + 1)y \, dy = (y + 1)e^x \, dx$ (*ii*) $\tan y \, dx + \sec^2 y \tan x \, dy = 0$

(*iii*) $\sqrt{1+x^2}\, dy + \sqrt{1+y^2}\, dx = 0$ (*iv*) $\sin^3 x = \sin y \dfrac{dy}{dx}$

7. (*i*) $\frac{dy}{dx} = e^{x+y} + e^{-x+y}$ (*ii*) $(xy^2 + 2x)dx + (x^2y + 2y)dy = 0$

(*iii*) $\frac{dy}{dx} = (\cos^2 x - \sin^2 x)\cos^2 y$ (*iv*) $xy\frac{dy}{dx} = 1 + x + y + xy$

8. (*i*) $y(1 + e^x)dy = (y + 1)e^x\, dx$ (*ii*) $\frac{dy}{dx} = y^2 \tan 2x$

(*iii*) $e^x\sqrt{1-y^2}\; dx + \frac{y}{x}\, dy = 0$ (*iv*) $(1 + e^{2x})dy + (1 + y^2)e^x dx = 0$

Solve the following initial value problems (Q. No. 9–12):

9. (*i*) $y' = 2e^x y^3$, $y(0) = 1/2$ (*ii*) $y' = -4xy^2$, $y(0) = 1$

(*iii*) $xyy' = y + 2$, $y(2) = 0$ (*iv*) $y' = y \cot 2x$, $y(\pi/4) = 2$

10. (*i*) $x(1 + y^2)dx - y(1 + x^2)dy = 0$, $y(0) = 1$

(*ii*) $(1 + y^2)(1 + \log x)dx + x\, dy = 0$, $y(1) = 1$

(*iii*) $(1 - y^2)(1 + \log x)dx + 2xy\, dy = 0$, $y(1) = 0$

(*iv*) $\sec^2 y\, (1 + x^2)dy + 2x \tan y\, dx = 0$, $y(1) = \pi/4$

11. (*i*) $\left(\frac{2 + \sin x}{1 + y}\right)\frac{dy}{dx} = -\cos x$, $y(0) = 1$ (*ii*) $(x + 1)\frac{dy}{dx} = 2e^{-y} - 1$, $y(0) = 0$.

(*iii*) $(1 + e^{2x})dy + (1 + y^2)e^x\, dx = 0$, $y(0) = 1$ (*iv*) $x(x^2 - 1)\frac{dy}{dx} = 1$, $y(2) = 0$

12. (*i*) $\frac{dy}{dx} = y \tan x$, $y(0) = 1$ (*ii*) $\log\left(\frac{dy}{dx}\right) = 3x + 4y$, $y(0) = 0$

(*iii*) $xy\frac{dy}{dx} = (x + 2)(y + 2)$, $y(1) = -1$ (*iv*) $(x^2 - yx^2)dy + (y^2 + x^2y^2)\, dx = 0$, $y(1) = 1$

Answers

1. (*i*) $y = \log(x^4y^2) + C$ (*ii*) $\sin^{-1} x + \sin^{-1} y = C$

(*iii*) $\log |y| = e^x + x + C$ (*iv*) $2e^{2y} = x^4 + C$

2. (*i*) $x^{-4} + y^{-4} = C$ (*ii*) $x^2 + y^2 + 2x - 4y + C = 0$

(*iii*) $\tan^{-1} y = \tan^{-1} x + C$ (*iv*) $y = e^{Cx}$

3. (*i*) $(1 - x^2)(1 + y^2) = C$ (*ii*) $\log |xy| + x - y = C$

(*iii*) $(a + x)(1 - ay) = Cy$ (*iv*) $-\frac{1 + \log y}{y} + (2 - x^2)\cos x + 2x \sin x = C$

4. (*i*) $\tan^{-1} x + \tan^{-1} y + \frac{1}{2}\log(1 + x^2)(1 + y^2) = C$

(*ii*) $x - \log|y| - \left(\frac{1}{x} + \frac{1}{y}\right) = C$ (*iii*) $(x + a)(1 - ay) = Cy$

(*iv*) $\sqrt{1-x^2} + \sqrt{1-y^2} = C$

5. (*i*) $\log |y + 1| = \frac{1}{2}x^2 + x + C$ (*ii*) $\tan y = C(1 - e^x)^{-2}$

(*iii*) $\frac{1}{2}(1 + \log x)^2 + \tan^{-1} y = C$ (*iv*) $\sin y = C \cos x$

6. (*i*) $y - \log |y + 1| = \log(e^x + 1) + C$ (*ii*) $\sin x \tan y = C$

(*iii*) $(x + \sqrt{1+x^2})(y + \sqrt{1+y^2}) = C$ (*iv*) $\cos y + \frac{1}{3}\cos^3 x - \cos x = C$

7. (*i*) $e^{-x} - e^{-y} = e^x + C$ (*ii*) $(x^2 + 2)(y^2 + 2) = C$

(*iii*) $\tan y = \frac{1}{2}\sin 2x + C$ (*iv*) $y = x + \log |x(1 + y)| + C$

8. (*i*) $y - \log |1 + y| = \log C(1 + e^x)$ (*ii*) $-\frac{1}{y} = \frac{1}{2}\log|\sec 2x| + C$

(*iii*) $(x - 1)e^x - \sqrt{1-y^2} = C$ (*iv*) $y + e^x = C(1 - ye^x)$

NOTES

9. (*i*) $y^2(8 - 4e^x) = 1$ (*ii*) $y = \dfrac{1}{2x^2 + 1}$

(*iii*) $y = 2 \log |y + 2| + \log \dfrac{|x|}{8}$ (*iv*) $y^2 = 4 \sin 2x$

10. (*i*) $y^2 - 2x^2 = 1$ (*ii*) $2(1 + \log x)^2 + 4 \tan^{-1} y = 2 + \pi$

(*iii*) $(1 + \log x)^2 = 2 \log |1 - y^2| + 1$ (*iv*) $(1 + x^2) \tan y = 2$

11. (*i*) $(2 + \sin x)(1 + y) = 4$ (*ii*) $(x + 1)(2 - e^y) = 1$

(*iii*) $\tan^{-1} y + \tan^{-1} e^x = \pi/2$ (*iv*) $y = \dfrac{1}{2} \log \dfrac{4|x^2 - 1|}{3x^2}$

12. (*i*) $y = \sec x$ (*ii*) $4e^{3x} + 3e^{-4y} = 7$

(*iii*) $y - x + 2 = 2 \log |x(y + 2)|$ (*iv*) $x = x^{-1} + y^{-1} + \log |y| - 1.$

Solution of $\dfrac{dy}{dx} = f(ax + by + c)$ by the Method of Separation of Variables

Consider the differential equation $\dfrac{dy}{dx} = f(ax + by + c)$, ...(1)

where $f(ax + by + c)$ is some function of '$ax + by + c$'.

Let $z = ax + by + c$. $\therefore \dfrac{dz}{dx} = a + b\dfrac{dy}{dx}$ or $\dfrac{dy}{dx} = \dfrac{\frac{dz}{dx} - a}{b}$

$\therefore$ (1) $\Rightarrow$ $\dfrac{\frac{dz}{dx} - a}{b} = f(z)$ $\Rightarrow$ $\dfrac{dz}{dx} = bf(z) + a$

$\Rightarrow$ $\dfrac{dz}{bf(z) + a} = dx$ ...(2)

In the differential equation (2), the variables x and z are separated.

Integrating (2), we get

$$\int \frac{dz}{bf(z) + a} = \int 1 \cdot dx + C.$$

$$\Rightarrow \quad \int \frac{dz}{bf(z) + a} = x + C, \quad \text{where } z = ax + by + c.$$

This represents the general solution of the differential equation (1).

Working Steps for Solving $\dfrac{dy}{dx} = f(ax + by + c)$

Step I. Identify the function $f(ax + by + c)$.

Step II. Put $z = ax + by + c$ and differentiate it w.r.t. x. Solve this to find the value of $\dfrac{dy}{dx}$.

Step III. Put the values of $\dfrac{dy}{dx}$ and $ax + by + c$ in the given differential equation. Separate the variables z and x and integrate both sides.

Step IV. Replace the value of z. This gives the general solution of the given differential equation.

SOLVED EXAMPLES

NOTES

Example 27. *Solve the differential equations:*

(*i*) $\frac{dy}{dx}\cos(x+y) = 1$ (*ii*) $\cos^2(x-2y) = 1 - 2\frac{dy}{dx}$.

Solution. (*i*) We have $\frac{dy}{dx}\cdot\cos(x+y) = 1$ or $\frac{dy}{dx} = \sec(x+y)$. ...(1)

RHS of (1) is a function of $x + y$.

$$z = x + y \Rightarrow \frac{dz}{dx} = 1 + \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{dz}{dx} - 1$$

$$\therefore \quad (1) \Rightarrow \frac{dz}{dx} - 1 = \sec z \Rightarrow \frac{dz}{dx} = 1 + \sec z$$

$$\Rightarrow \frac{dz}{1+\sec z} = dx \qquad \text{(Variables are separate)}$$

$$\Rightarrow \frac{\cos z}{1+\cos z}\,dz = dx \Rightarrow \left(1 - \frac{1}{1+\cos z}\right)dz = dx$$

Integrating, we get

$$\int\left(1 - \frac{1}{1+\cos z}\right)dz = \int 1\cdot dx + \text{C}.$$

$$\Rightarrow z - \int \frac{1}{2\cos^2\frac{z}{2}}\,dz = x + \text{C}$$

$$\Rightarrow z - \frac{1}{2}\int \sec^2\frac{z}{2}\,dz = x + \text{C}$$

$$\Rightarrow z - \frac{1}{2}\,\frac{\tan\frac{z}{2}}{1/2} = x + \text{C} \Rightarrow x + y - \tan\frac{x+y}{2} = x + \text{C}$$

$$\Rightarrow \mathbf{y = \tan\frac{x+y}{2} + C}.$$

(*ii*) We have $\cos^2(x-2y) = 1 - 2\frac{dy}{dx}$.

$$\therefore \quad \frac{dy}{dx} = \frac{1-\cos^2(x-2y)}{2} = \frac{\sin^2(x-2y)}{2}$$

$$= \frac{1}{2}\left(\frac{1-\cos(2x-4y)}{2}\right) = \frac{1}{4}(1-\cos(2x-4y))$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{4}(1-\cos(2x-4y)) \qquad ...(1)$$

RHS of (1) is a function of $2x - 4y$.

$$z = 2x - 4y \Rightarrow \frac{dz}{dx} = 2 - 4\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{4}\left(2 - \frac{dz}{dx}\right)$$

$$\therefore \quad (1) \Rightarrow \frac{1}{4}\left(2 - \frac{dz}{dx}\right) = \frac{1}{4}(1-\cos z)$$

NOTES

$$\Rightarrow \quad 2 - \frac{dz}{dx} = 1 - \cos z \quad \Rightarrow \quad \frac{dz}{dx} = 1 + \cos z$$

$$\Rightarrow \quad \frac{dz}{1+\cos z} = dx \qquad \text{(Variables are separate)}$$

Integrating, we get

$$\int \frac{dz}{1+\cos z} = \int 1 \cdot dx + C.$$

$$\Rightarrow \quad \int \frac{dz}{2\cos^2 \frac{z}{2}} = x + C \quad \Rightarrow \quad \frac{1}{2}\int \sec^2 \frac{z}{2}\, dz = x + C$$

$$\Rightarrow \quad \frac{1}{2} \cdot \frac{\tan \frac{z}{2}}{1/2} = x + C \quad \Rightarrow \quad \tan \frac{2x - 4y}{2} = x + C$$

$$\Rightarrow \quad \mathbf{\tan (x - 2y) = x + C}.$$

Example 28. *Solve the differential equations:*

(*i*) $\frac{dy}{dx} = \frac{x + 2y - 1}{x + 2y + 1}$ (*ii*) $\frac{dy}{dx} = \frac{2x - y + 2}{2y - 4x + 1}$.

Solution. (*i*) We have $\frac{dy}{dx} = \frac{(x + 2y) - 1}{(x + 2y) + 1}$. ...(1)

RHS of (1) is a function of $x + 2y$.

Let $z = x + 2y$. $\therefore$ $\frac{dz}{dx} = 1 + 2\frac{dy}{dx}$ or $\frac{dy}{dx} = \frac{1}{2}\left(\frac{dz}{dx} - 1\right)$

$$\therefore \quad (1) \Rightarrow \quad \frac{1}{2}\left(\frac{dz}{dx} - 1\right) = \frac{z-1}{z+1} \quad \Rightarrow \quad \frac{dz}{dx} - 1 = \frac{2z-2}{z+1}$$

$$\Rightarrow \quad \frac{dz}{dx} = \frac{2z-2}{z+1} + 1 \quad \Rightarrow \quad \frac{dz}{dx} = \frac{3z-1}{z+1}$$

$$\Rightarrow \quad \frac{z+1}{3z-1}\, dz = dx \qquad \text{(Variables are separate)}$$

Integrating, we get $\int \frac{z+1}{3z-1}\, dz = \int 1.dx + C$

$$\Rightarrow \quad \frac{1}{3}\int \frac{3z - 1 + 4}{3z - 1}\, dz = x + C \quad \Rightarrow \quad \frac{1}{3}\int \left(1 + \frac{4}{3z-1}\right) dz = x + C$$

$$\Rightarrow \quad \frac{1}{3}\left(z + \frac{4}{3}\log|3z - 1|\right) = x + C \quad \Rightarrow \quad 3z + 4\log|3z - 1| = 9x + 9C$$

$$\Rightarrow \quad 3(x + 2y) + 4\log|3(x + 2y) - 1| = 9x + C_1, \quad \text{where } C_1 = 9C$$

$$\Rightarrow \quad \mathbf{6(y - x) + 4\log|3x + 6y - 1| = C_1}.$$

(*ii*) We have $\frac{dy}{dx} = \frac{2x - y + 2}{2y - 4x + 1}$ *i.e.*, $\frac{dy}{dx} = \frac{(2x - y) + 2}{-2(2x - y) + 1}$. ...(1)

RHS of (1) is a function of $2x - y$.

NOTES

Let $z = 2x - y$. $\therefore \dfrac{dz}{dx} = 2 - \dfrac{dy}{dx}$ or $\dfrac{dy}{dx} = 2 - \dfrac{dz}{dx}$

$$(1) \Rightarrow \quad 2 - \frac{dz}{dx} = \frac{z+2}{-2z+1} \Rightarrow \frac{dz}{dx} = 2 - \frac{z+2}{-2z+1} = \frac{-5z}{-2z+1}$$

$$\Rightarrow \quad \frac{dz}{dx} = \frac{5z}{2z-1} \Rightarrow \frac{2z-1}{5z}\, dz = dx \qquad \text{(Variables are separate)}$$

Integrating, we get $\displaystyle\int \frac{2z-1}{5z}\, dz = \int dx + \text{C}.$

$$\Rightarrow \quad \int\left(\frac{2}{5} - \frac{1}{5z}\right) dz = x + \text{C} \Rightarrow \frac{2}{5}\, z - \frac{1}{5} \log|z| = x + \text{C}$$

$$\Rightarrow \quad 2z - \log|z| = 5x + 5\text{C}$$

$$\Rightarrow \quad 2(2x - y) - \log|2x - y| = 5x + \text{C}_1, \text{ where } \text{C}_1 = 5\text{C}$$

$$\Rightarrow \quad \mathbf{x + 2y + \log|2x - y| + C_1 = 0}.$$

EXERCISE H

Find the general solution of the following differential equations (Q. No. 1–4):

1. (*i*) $\dfrac{dy}{dx} = \dfrac{2}{x+2y-3}$ (*ii*) $(x + y + 1)\dfrac{dy}{dx} = 1$

2. (*i*) $\dfrac{dy}{dx} = (3x + 2y + 1)^2$ (*ii*) $(x + y)^2\dfrac{dy}{dx} = k^2$

3. (*i*) $\dfrac{dy}{dx} = \dfrac{x+y+1}{x+y}$ (*ii*) $\dfrac{dy}{dx} = \dfrac{x-2y+3}{2x-4y+5}$

4. (*i*) $\dfrac{dy}{dx} = \dfrac{x+y+1}{2x+2y+3}$ (*ii*) $\dfrac{dy}{dx} = \dfrac{x+2y+1}{2x+4y+3}$.

5. Solve the following initial value problems:

(*i*) $(x + y + 1)^2\, dy = dx,\ y(-1) = 0$ (*ii*) $\cos(x + y)dy = dx,\ y(0) = 0.$

Answers

1. (*i*) $2y = 4 \log|x + 2y + 1| + \text{C}$ (*ii*) $y = \log|x + y + 2| + \text{C}$

2. (*i*) $\sqrt{2}\,(3x + 2y + 1) = \sqrt{3}\tan[\sqrt{6}\,(x + \text{C})]$

(*ii*) $y = k \tan^{-1}\dfrac{x+y}{k} + \text{C}$

3. (*i*) $2(y - x) = \log|2x + 2y + 1| + \text{C}$ (*ii*) $x^2 - 4xy + 4y^2 + 6x - 10y = \text{C}$

4. (*i*) $3(2y - x) + \log|3x + 3y + 4| = \text{C}$ (*ii*) $4(2y - x) + \log|4x + 8y + 5| = \text{C}$

5. (*i*) $\tan y = x + y + 1$ (*ii*) $y = \tan\dfrac{x+y}{2}$.

HOMOGENEOUS DIFFERENTIAL EQUATIONS AND THEIR SOLUTION

Homogeneous Function

A function $f(x, y)$ of x and y is called a **homogeneous function** if $f(\lambda x, \lambda y) = \lambda^n f(x, y)$. The number n is called the **degree** of the homogeneous function $f(x, y)$.

NOTES

Illustration. Let $f(x, y) = x^3 + 2xy^2 - 3y^3$.

$\therefore \quad f(\lambda x, \lambda y) = (\lambda x)^3 + 2(\lambda x)(\lambda y)^2 - 3(\lambda y)^3 = \lambda^3[x^3 + 2xy^2 - 3y^3] = \lambda^3 f(x, y)$

$\therefore \quad x^3 + 2xy^2 - 3y^3$ is a homogeneous function of degree 3.

If $f(x, y)$ is a homogeneous function of degree n, then $f(x, y)$ can be expressed as $x^n \phi\left(\frac{y}{x}\right)$, where $\phi\left(\frac{y}{x}\right)$ is some function of $\frac{y}{x}$.

Illustration $f(x, y) = x^2 + 7xy - 3y^2$ is a homogeneous function of degree 2 and we have $\quad f(x, y) = x^2 + 7xy - 3y^2 = x^2\left[1 + 7\left(\frac{y}{x}\right) - 3\left(\frac{y}{x}\right)^2\right]$

and $1 + 7\left(\frac{y}{x}\right) - 3\left(\frac{y}{x}\right)^2$ is a function of $\frac{y}{x}$.

Homogeneous Differential Equation

If $f(x, y)$ and $g(x, y)$ are homogeneous functions of same degree then the differential equation

$$\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$$

is called a **homogeneous differential equation**.

Let $f(x, y)$ and $g(x, y)$ be homogeneous functions of degree n each.

$\therefore \quad f(x, y) = x^n F(y/x)$ and $g(x, y) = x^n G(y/x)$ for some functions $F(y/x)$ and $G(y/x)$ of y/x.

$\therefore \quad \frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$ becomes $\frac{dy}{dx} = \frac{x^n F(y/x)}{x^n G(y/x)} = \frac{F(y/x)}{G(y/x)} = \phi(y/x)$, say

$\therefore \quad$ A homogeneous differential equation can also be expressed as

$$\frac{dy}{dx} = \phi(y/x).$$

Illustration. Let $\frac{dy}{dx} = \frac{x^3 - 2y^3}{xy^2 + 7y^3}$ ...(1)

(1) is a homogeneous differential equation, because $x^3 - 2y^3$ and $xy^2 + 7y^3$ are homogeneous functions of degree 3 each.

(1) can also be expressed as

$$\frac{dy}{dx} = \frac{1 - 2(y/x)^3}{(y/x)^2 + 7(y/x)^3}.$$

Solution of Homogeneous Differential Equation

Let $$\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)} \quad ...(1)$$

be a homogeneous differential equation.

$\therefore \quad f(x, y)$ and $g(x, y)$ are homogeneous functions of same degree, say, n.

Let $\quad f(x, y) = x^n F(y/x)$ and $\quad g(x, y) = x^n G(y/x)$

for some functions $F(y/x)$ and $G(y/x)$ of y/x.

$$\therefore \quad \frac{f(x, y)}{g(x, y)} = \frac{x^n F(y/x)}{x^n G(y/x)} = \frac{F(y/x)}{G(y/x)} = \phi(y/x), \text{ say}$$

$\therefore$ (1) $\Rightarrow$ $\dfrac{dy}{dx} = \phi(y/x)$ ...(2)

Let $y = vx$. $\therefore$ $\dfrac{dy}{dx} = v(1) + x\dfrac{dv}{dx} = v + x\dfrac{dv}{dx}$

$\therefore$ (2) $\Rightarrow$ $v + x\dfrac{dv}{dx} = \phi(v)$ $\Rightarrow$ $\dfrac{dv}{\phi(v) - v} = \dfrac{dx}{x}$ (Variables are separate)

Integrating both sides, we get $\displaystyle\int \frac{dv}{\phi(v) - v} = \int \frac{dx}{x} + C.$

$$\Rightarrow \quad \boldsymbol{\int \frac{dv}{\phi(v) - v} = \log|x| + C, \text{ where } v = \frac{y}{x}}.$$

This equation is solved and v is replaced by y/x.

Working Steps for Solving $\dfrac{dy}{dx} = \phi\left(\dfrac{y}{x}\right)$

Step I. Make sure that R.H.S. is either a function of 'y/x' or the quotient of two homogeneous functions of 'same' degree.

Step II. Put y = vx and differentiate it w.r.t. x to get $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$.

Step III. Put the values of $\dfrac{dy}{dx}$ and y in the given differential equation. Separate the variables v and x and integrate both sides.

Step IV. Replace the value of v. This gives the general solution of the given differential equation.

SOLVED EXAMPLES

Example 29. *Solve:* $y' = \dfrac{x + y}{x}$.

Solution. We have $\dfrac{dy}{dx} = \dfrac{x + y}{x}$. ...(1)

This is a homogeneous differential equation. $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$\therefore$ (1) $\Rightarrow$ $v + x\dfrac{dv}{dx} = \dfrac{x + vx}{x} = 1 + v$

$\Rightarrow$ $x\dfrac{dv}{dx} = 1 \Rightarrow dv = \dfrac{dx}{x}$ (Variables are separate)

Integrating, we get $\displaystyle\int 1 \cdot dv = \int \frac{dx}{x} + C.$

$\Rightarrow$ $v = \log|x| + C \Rightarrow$ $\boldsymbol{\dfrac{y}{x} = \log|x| + C}$.

Example 30. *Solve:* $x^2\dfrac{dy}{dx} = x^2 + xy + y^2$.

Solution. We have $x^2\dfrac{dy}{dx} = x^2 + xy + y^2$. $\therefore$ $\dfrac{dy}{dx} = \dfrac{x^2 + xy + y^2}{x^2}$ ...(1)

This is a homogeneous differential equation. $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

NOTES

$\therefore$ (1) $\Rightarrow$ $v + x\dfrac{dv}{dx} = \dfrac{x^2 + x(vx) + (vx)^2}{x^2} = 1 + v + v^2$

$\Rightarrow$ $x\dfrac{dv}{dx} = 1 + v^2 \Rightarrow \dfrac{dv}{1+v^2} = \dfrac{dx}{x}$ (Variables are separate)

Integrating, we get $\displaystyle\int \frac{dv}{1+v^2} = \int \frac{dx}{x} + C$

$\Rightarrow$ $\tan^{-1} v = \log |x| + C \Rightarrow \mathbf{\tan^{-1} \dfrac{y}{x} = \log |x| + C}$.

Example 31. *Solve:* $2xy \dfrac{dy}{dx} = x^2 + y^2$.

Solution. We have $\dfrac{dy}{dx} = \dfrac{x^2 + y^2}{2xy}$. ...(1)

This is a homogeneous differential equation. $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$\therefore$ (1) $\Rightarrow$ $v + x\dfrac{dv}{dx} = \dfrac{x^2 + (vx)^2}{2x(vx)} = \dfrac{1+v^2}{2v}$

$\Rightarrow$ $x\dfrac{dv}{dx} = \dfrac{1+v^2}{2v} - v = \dfrac{1-v^2}{2v}$

$\Rightarrow$ $\dfrac{2v}{1-v^2}\, dv = \dfrac{dx}{x}$ (Variables are separate)

$\Rightarrow$ $\dfrac{2v}{v^2-1}\, dv + \dfrac{dx}{x} = 0$

$\Rightarrow$ $\log |v^2 - 1| + \log |x| = \log C$

$\Rightarrow$ $\log |x(v^2 - 1)| = \log C$

$\Rightarrow$ $x(v^2 - 1) = \pm C \Rightarrow x\left(\dfrac{y^2}{x^2} - 1\right) = C'$, where $C' = \pm C$

$\Rightarrow$ $\mathbf{y^2 - x^2 = C'x}$.

Example 32. *Solve:* $x\,dy - y\,dx = \sqrt{x^2 + y^2}\, dx$.

Solution. We have $x\dfrac{dy}{dx} - y = \sqrt{x^2 + y^2}$. $\therefore$ $\dfrac{dy}{dx} = \dfrac{y}{x} + \sqrt{1 + \left(\dfrac{y}{x}\right)^2}$ ...(1)

This is a homogeneous differential equation. $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$\therefore$ (1) $\Rightarrow$ $v + x\dfrac{dv}{dx} = v + \sqrt{1+v^2} \Rightarrow \dfrac{dv}{\sqrt{1+v^2}} = \dfrac{dx}{x}$

(Variables are separate)

$\Rightarrow$ $\displaystyle\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{dx}{x} + \log C$

$\Rightarrow$ $\log |v + \sqrt{1+v^2}| = \log |x| + \log C$

NOTES

$$\Rightarrow \quad \log\left|\frac{y}{x}+\sqrt{1+\frac{y^2}{x^2}}\right| = \log(C\,|x|) \quad \Rightarrow \quad \left|\frac{y+\sqrt{x^2+y^2}}{x}\right| = C\,|x|$$

$$\Rightarrow \quad |\,y+\sqrt{x^2+y^2}\,| = Cx^2 \quad \Rightarrow \quad y+\sqrt{x^2+y^2} = \pm Cx^2$$

$$\Rightarrow \quad \mathbf{y+\sqrt{x^2+y^2} = C_1x^2}. \qquad \text{(Putting } C_1 = \pm C)$$

Example 33. *Solve:* $x^2\frac{dy}{dx} = 2xy + y^2$.

Solution. We have $x^2\frac{dy}{dx} = 2xy + y^2$. $\therefore \quad \frac{dy}{dx} = \frac{2xy+y^2}{x^2}$ ...(1)

This is a homogeneous differential equation. $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\therefore \quad (1) \quad \Rightarrow \quad v + x\frac{dv}{dx} = \frac{2x(vx)+(vx)^2}{x^2} = 2v + v^2$$

$$\Rightarrow \quad x\frac{dv}{dx} = (2v+v^2) - v = v + v^2 \quad \Rightarrow \quad \frac{dv}{v(1+v)} = \frac{dx}{x}$$

(Variables are separate)

Integrating, we get $\int \frac{dv}{v(1+v)} = \int \frac{dx}{x} + \log C.$

$$\int\left(\frac{1}{v}-\frac{1}{1+v}\right)dv = \log\,|x| + \log C$$

$$\Rightarrow \quad \log\,|v| - \log\,|1+v| = \log C\,|x|$$

$$\Rightarrow \quad \log\left|\frac{v}{1+v}\right| = \log C\,|x| \quad \Rightarrow \quad \left|\frac{y/x}{1+y/x}\right| = C\,|x|$$

$$\Rightarrow \quad \left|\frac{y}{x+y}\right| = C\,|x| \quad \Rightarrow \quad |y| = C\,|x(x+y)|$$

$$\Rightarrow \quad y = \pm Cx(x+y)$$

$$\Rightarrow \quad \mathbf{y = C_1x(x+y), \quad \text{where } C_1 = \pm C}.$$

Example 34. *Solve:* $\frac{dy}{dx} = \frac{y}{x} + \sin\frac{y}{x}$.

Solution. We have $\frac{dy}{dx} = \frac{y}{x} + \sin\frac{y}{x}$. ...(1)

This is a homogeneous differential equation. $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\therefore \quad (1) \quad \Rightarrow \quad v + x\frac{dv}{dx} = v + \sin v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \sin v \Rightarrow \operatorname{cosec} v\,dv = \frac{dx}{x} \qquad \text{(Variables are separate)}$$

Integrating, we get $\int \operatorname{cosec} v\,dv = \int \frac{dx}{x} + \log C.$

$$\Rightarrow \quad \log\left|\tan\frac{v}{2}\right| = \log\,|x| + \log C$$

$$\Rightarrow \quad \log\left|\tan\frac{y}{2x}\right| = \log C\,|x|$$

NOTES

$$\Rightarrow \quad \left|\tan \frac{y}{2x}\right| = C\,|x| \quad \Rightarrow \quad \tan \frac{y}{2x} = \pm Cx$$

$$\Rightarrow \quad \mathbf{\tan \frac{y}{2x} = C_1 x}. \qquad \text{(Putting } C_1 = \pm C\text{)}$$

Example 35. *Solve:* $x \frac{dy}{dx} = y - x \tan \frac{y}{x}$.

Solution. We have $x \frac{dy}{dx} = y - x \tan \frac{y}{x}$.

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} - \tan \frac{y}{x} \qquad \text{...(1)}$$

This is a homogeneous differential equation. $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\therefore \quad (1) \quad \Rightarrow \quad v + x\frac{dv}{dx} = v - \tan v$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\tan v \quad \Rightarrow \quad \cot v\, dv = -\frac{dx}{x}$$

(Variables are separate)

Integrating, we get $\int \cot v\, dv = -\int \frac{dx}{x} + C$.

$$\Rightarrow \quad \log\,|\sin v| = -\log\,|x| + \log C \quad \Rightarrow \quad \log\,|\sin v| = \log \frac{C}{|x|}$$

$$\Rightarrow \quad |\sin v| = \frac{C}{|x|} \quad \Rightarrow \quad |x|\left|\sin \frac{y}{x}\right| = C$$

$$\Rightarrow \quad \left|x \sin \frac{y}{x}\right| = C \quad \Rightarrow \quad x \sin \frac{y}{x} = \pm C$$

$$\Rightarrow \quad \mathbf{x \sin \frac{y}{x} = C_1}. \qquad \text{(Putting } C_1 = \pm C\text{)}$$

Example 36. *Solve:* $(x^3 - 3xy^2)dx = (y^3 - 3x^2y)dy$.

Solution. We have $(x^3 - 3xy^2)dx = (y^3 - 3x^2y)dy$ *i.e.,* $\frac{dy}{dx} = \frac{x^3 - 3xy^2}{y^3 - 3x^2y}$. ...(1)

This is a homogeneous differential equation. $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\therefore \quad (1) \quad \Rightarrow \quad v + x\frac{dv}{dx} = \frac{x^3 - 3x(vx)^2}{(vx)^3 - 3x^2(vx)} = \frac{x^3 - 3v^2x^3}{v^3x^3 - 3vx^3} = \frac{1 - 3v^2}{v^3 - 3v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1 - 3v^2}{v^3 - 3v} - v = \frac{1 - 3v^2 - v^4 + 3v^2}{v^3 - 3v} = \frac{1 - v^4}{v^3 - 3v}$$

$$\Rightarrow \quad \frac{v^3 - 3v}{1 - v^4}\, dv = \frac{dx}{x} \qquad \text{(Variables are separate)}$$

Integrating, we get $\int \frac{v^3 - 3v}{1 - v^4}\, dv = \int \frac{dx}{x} + \log C$.

$$\Rightarrow \quad \int \frac{v^3}{1 - v^4}\, dv - 3\int \frac{v}{1 - v^4}\, dv = \log\,|x| + \log C$$

NOTES

$$\Rightarrow \quad -\frac{1}{4}\int \frac{-4v^3}{1-v^4}\,dv - \frac{3}{2}\int \frac{2v}{1-v^4}dv = \log C\,|x|$$

$$\Rightarrow \quad -\frac{1}{4}\log|1-v^4| - \frac{3}{2}\int \frac{dt}{1-t^2} = \log C\,|x|, \text{ where } t = v^2$$

$$\Rightarrow \quad -\frac{1}{4}\log|1-v^4| - \frac{3}{2}\cdot\frac{1}{2(1)}\log\left|\frac{1+t}{1-t}\right| = \log C\,|x|$$

$$\Rightarrow \quad -\frac{1}{4}\log|1-v^4| - \frac{3}{4}\log\left|\frac{1+v^2}{1-v^2}\right| = \log C\,|x|$$

$$\Rightarrow \quad \log\left|(1-v^4)\,.\,\frac{(1+v^2)^3}{(1-v^2)^3}\right| = \log(C\,|x|)^{-4} \quad \Rightarrow \quad \frac{(1+v^2)^4}{(1-v^2)^2} = \frac{1}{C^4x^4}$$

$$\Rightarrow \quad C^4x^4\,(1+v^2)^4 = (1-v^2)^2 \quad \Rightarrow \quad C^4x^4\left(1+\frac{y^2}{x^2}\right)^4 = \left(1-\frac{y^2}{x^2}\right)^2$$

$$\Rightarrow \quad C^4x^4\,.\,\frac{(x^2+y^2)^4}{x^8} = \frac{(x^2-y^2)^2}{x^4}$$

$$\Rightarrow \quad \mathbf{C_1(x^2+y^2)^4 = (x^2-y^2)^2, \text{ where } C_1 = C^4.}$$

Example 37. *Solve:* $y\left(x\cos\frac{y}{x} + y\sin\frac{y}{x}\right)dx = x\left(y\sin\frac{y}{x} - x\cos\frac{y}{x}\right)dy.$

Solution. The given equation is $\dfrac{dy}{dx} = \dfrac{y\left(x\cos\frac{y}{x} + y\sin\frac{y}{x}\right)}{x\left(y\sin\frac{y}{x} - x\cos\frac{y}{x}\right)}.$ ...(1)

This is a homogeneous differential equation. $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$$\therefore \quad (1) \quad \Rightarrow \quad v + x\frac{dv}{dx} = \frac{vx\,(x\cos v + vx\sin v)}{x\,(vx\sin v - x\cos v)} = \frac{v\,(\cos v + v\sin v)}{v\sin v - \cos v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v\,(\cos v + v\sin v)}{v\sin v - \cos v} - v = \frac{2v\cos v}{v\sin v - \cos v}$$

$$\Rightarrow \quad \frac{v\sin v - \cos v}{v\cos v} = 2\,\frac{dx}{x} \qquad \text{(Variables are separate)}$$

Integrating, we get $\displaystyle\int \frac{v\sin v - \cos v}{v\cos v}\,dv = 2\int\frac{dx}{x} + \log C.$

$$\Rightarrow \quad -\int\frac{\cos v - v\sin v}{v\cos v}\,dv = 2\log|x| + \log C$$

$$\Rightarrow \quad -\log|v\cos v| = \log Cx^2$$

$$\Rightarrow \quad \frac{1}{v\cos v} = \pm Cx^2 \quad \Rightarrow \quad \frac{x}{y\cos(y/x)} = \pm Cx^2$$

$$\Rightarrow \quad xy\cos\left(\frac{y}{x}\right) = \pm\frac{1}{C} \quad \Rightarrow \mathbf{xy\cos\left(\frac{y}{x}\right) = C_1, \text{ where } C_1 = \pm\frac{1}{C}}.$$

Note. The above question can also be given as follows:

Solve: $y(x\,dy - y\,dx)\sin\frac{y}{x} = x(y\,dx + x\,dy)\cos\frac{y}{x}.$

NOTES

Example 38. *Check whether the following differential equation is homogeneous or not:* $x^2 \frac{dy}{dx} - xy = 1 + \cos\left(\frac{y}{x}\right), x \neq 0$?

Find the general solution of the differential equation using substitution $y = vx$.

Solution. We have

$$x^2 \frac{dy}{dx} - xy = 1 + \cos\left(\frac{y}{x}\right), x \neq 0.$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{xy + 1 + \cos\left(\frac{y}{x}\right)}{x^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} + \frac{1}{x^2}\left(1 + \cos\left(\frac{y}{x}\right)\right) \quad ...(1)$$

This is not a homogeneous differential equation, because RHS is not a function of y/x.

Let $\quad y = vx \quad \therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\therefore \quad (1) \Rightarrow \quad v + x\frac{dv}{dx} = v + \frac{1}{x^2}(1 + \cos v)$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{1 + \cos v}{x^3} \Rightarrow \frac{dv}{1 + \cos v} = \frac{dx}{x^3}$$

(Variables are separate)

Integrating, we get

$$\int \frac{dv}{1 + \cos v} = \int \frac{dx}{x^3} + \text{C}.$$

$$\Rightarrow \quad \int \frac{1 - \cos v}{\sin^2 v}\, dv = \frac{x^{-2}}{-2} + \text{C}$$

$$\Rightarrow \quad \int (\text{cosec}^2 v - \cot v \,\text{cosec}\, v)\, dv = -\frac{1}{2x^2} + \text{C}$$

$$\Rightarrow \quad -\cot v + \text{cosec}\, v = -\frac{1}{2x^2} + \text{C}$$

$$\Rightarrow \quad \mathbf{cosec}\,\frac{\boldsymbol{y}}{\boldsymbol{x}} - \mathbf{cot}\,\frac{\boldsymbol{y}}{\boldsymbol{x}} + \frac{\mathbf{1}}{\mathbf{2}\boldsymbol{x}^{\mathbf{2}}} = \mathbf{C}.$$

Example 39. *Solve:* $\left(x \sin^2\left(\frac{y}{x}\right) - y\right)dx + x\,dy = 0$ *given that* $y = \frac{\pi}{4}$ *when* $x = 1$.

Solution. We have $\left(x \sin^2\left(\frac{y}{x}\right) - y\right)dx + x\,dy = 0.$

$$\therefore \quad \frac{dy}{dx} = \frac{y - x\sin^2\left(\frac{y}{x}\right)}{x} \quad i.e., \quad \frac{dy}{dx} = \frac{y}{x} - \sin^2\left(\frac{y}{x}\right) \quad ...(1)$$

This is a homogeneous differential equation. $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$\therefore$ (1) $\Rightarrow$ $v + x\dfrac{dv}{dx} = \dfrac{vx}{x} - \sin^2\left(\dfrac{vx}{x}\right)$

$\Rightarrow$ $x\dfrac{dv}{dx} = -\sin^2 v \Rightarrow \text{cosec}^2 v\, dv = -\dfrac{dx}{x}$

(Variables are separate)

Integrating, we get $\displaystyle\int \text{cosec}^2 v\, dv = -\int \frac{dx}{x} + C$

$\Rightarrow$ $-\cot v = -\log|x| + C$

$\Rightarrow$ $\log|x| - \cot(y/x) = C$ ...(2)

Now, $y = \pi/4$ when $x = 1$.

$\therefore$ (2) $\Rightarrow$ $\log|1| - \cot((\pi/4)/1) = C \Rightarrow C = 0 - 1 = -1$

$\therefore$ (2) $\Rightarrow$ $\mathbf{\log|x| - \cot(y/x) = -1}$. This is the required solution.

Example 40. *Solve:* $x^2\, dy + (xy + y^2)dx = 0$, *given that* $y = 1$ *when* $x = 1$.

Solution. We have $x^2\, dy + y(x + y)\, dx = 0$. $\therefore$ $\dfrac{dy}{dx} = -\dfrac{y(x+y)}{x^2}$ ...(1)

This is a homogeneous differential equation. $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$\therefore$ (1) $\Rightarrow$ $v + x\dfrac{dv}{dx} = -\dfrac{vx(x+vx)}{x^2} = -v(1+v)$

$\Rightarrow$ $x\dfrac{dv}{dx} = -v - v^2 - v = -v(2+v)$

$\Rightarrow$ $\dfrac{dv}{v(2+v)} = -\dfrac{dx}{x}$ (Variables are separate)

Integrating, we get $\displaystyle\int \frac{dv}{v(2+v)} = -\int \frac{dx}{x} + \log C.$

$\Rightarrow$ $\displaystyle\int \left(\frac{1}{v(2)} + \frac{1}{(-2)(2+v)}\right) dv = -\log|x| + \log C$

$\Rightarrow$ $\dfrac{1}{2}\log|v| - \dfrac{1}{2}\log|2+v| = \log\dfrac{C}{|x|}$

$\Rightarrow$ $\dfrac{1}{2}\log\left|\dfrac{v}{2+v}\right| = \log\dfrac{C}{|x|} \Rightarrow \log\left|\dfrac{v}{2+v}\right| = 2\log\dfrac{C}{|x|}$

$\Rightarrow$ $\left|\dfrac{v}{2+v}\right| = \dfrac{C^2}{x^2} \Rightarrow \left|\dfrac{y/x}{2+y/x}\right| = \dfrac{C^2}{x^2}$

$\Rightarrow$ $\dfrac{y}{2x+y} = \pm\dfrac{C^2}{x^2}$

$\Rightarrow$ $x^2y = k(2x+y)$, where $k = \pm C^2$

Now, $y = 1$ when $x = 1$. $\therefore$ $(1)^2(1) = k(2(1) + 1)$ *i.e.,* $k = 1/3$

$\therefore$ The required solution is $x^2y = \dfrac{1}{3}(2x+y)$ or $\mathbf{3x^2y = 2x + y}$.

NOTES

Example 41. *Solve:* $\frac{dy}{dx}=\frac{x(2y-x)}{x(2y+x)}$, $y(1)=1$.

Solution. We have $\frac{dy}{dx}=\frac{x(2y-x)}{x(2y+x)}$ *i.e.*, $\frac{dy}{dx}=\frac{2y-x}{2y+x}$. ...(1)

This is a homogeneous differential equation. $y=vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

$$\therefore \quad (1) \Rightarrow \quad v+x\frac{dv}{dx}=\frac{2vx-x}{2vx+x}=\frac{2v-1}{2v+1}$$

$$\Rightarrow \quad x\frac{dv}{dx}=\frac{2v-1}{2v+1}-v=\frac{2v-1-2v^2-v}{2v+1}=-\frac{2v^2-v+1}{2v+1}$$

$$\Rightarrow \quad \frac{2v+1}{2v^2-v+1}\,dv=-\frac{dx}{x}$$ (Variables are separate)

Integrating, we get $\int \frac{2v+1}{2v^2-v+1}\,dv=-\int \frac{dx}{x}+C$.

$$\Rightarrow \quad \int \frac{2v+1}{2v^2-v+1}\,dv=-\log|x|+C$$ ...(1)

Now $\int \frac{2v+1}{2v^2-v+1}\,dv=\frac{1}{2}\int \frac{(4v-1)+3}{2v^2-v+1}\,dv$

$$=\frac{1}{2}\int \frac{4v-1}{2v^2-v+1}\,dv+\frac{3}{2}\int \frac{dv}{2v^2-v+1}$$

$$=\frac{1}{2}\log|2v^2-v+1|+\frac{3}{4}\int \frac{dv}{\left(v-\frac{1}{4}\right)^2+\frac{7}{16}}$$

$$=\frac{1}{2}\log|2v^2-v+1|+\frac{3}{4}\cdot\frac{1}{\sqrt{7}/4}\tan^{-1}\frac{v-\frac{1}{4}}{\sqrt{7}/4}$$

$$=\frac{1}{2}\log|2v^2-v+1|+\frac{3}{\sqrt{7}}\tan^{-1}\frac{4v-1}{\sqrt{7}}$$

$$\therefore \quad (1) \Rightarrow \quad \frac{1}{2}\log|2v^2-v+1|+\frac{3}{\sqrt{7}}\tan^{-1}\frac{4v-1}{\sqrt{7}}=-\log|x|+C$$

$$\Rightarrow \quad \frac{1}{2}\log\left|\frac{2y^2}{x^2}-\frac{y}{x}+1\right|+\frac{3}{\sqrt{7}}\tan^{-1}\frac{4y-x}{\sqrt{7}x}=-\log|x|+C$$

$$\Rightarrow \frac{1}{2}\log|2y^2-xy+x^2|-\log|x|+\frac{3}{\sqrt{7}}\tan^{-1}\frac{4y-x}{\sqrt{7}\,x}=-\log|x|+C$$

$$\Rightarrow \quad \frac{1}{2}\log|2y^2-xy+x^2|+\frac{3}{\sqrt{7}}\tan^{-1}\frac{4y-x}{\sqrt{7}x}=C$$ ...(2)

We have $y=1$ when $x=1$.

$$\therefore \quad (2) \Rightarrow \quad \frac{1}{2}\log|2-1+1|+\frac{3}{\sqrt{7}}\tan^{-1}\frac{4-1}{\sqrt{7}}=C$$

$$\therefore \qquad C = \frac{1}{2}\log 2 + \frac{3}{\sqrt{7}}\tan^{-1}\frac{3}{\sqrt{7}}$$

$\therefore$ Using (2), the required solution is

$$\mathbf{\frac{1}{2}\log|2y^2 - xy + x^2| + \frac{3}{\sqrt{7}}\tan^{-1}\frac{4y - x}{\sqrt{7}x} = \frac{1}{2}\log 2 + \frac{3}{\sqrt{7}}\tan^{-1}\frac{3}{\sqrt{7}}}.$$

NOTES

Solution of Homogeneous Differential Equation $\frac{dx}{dy} = y(x/y)$

We have $\qquad \frac{dx}{dy} = \psi(x/y)$. ...(1)

Let $x = vy$. $\quad \therefore \quad \frac{dx}{dy} = v(1) + y\frac{dv}{dy} = v + y\frac{dv}{dy}$

$\therefore$ (1) $\Rightarrow$ $\quad v + y\frac{dv}{dy} = \psi(v) \quad \Rightarrow \quad \frac{dv}{\psi(v) - v} = \frac{dy}{y}$ (Variables are separate)

Integrating both sides, we get $\int \frac{dv}{\psi(v) - v} = \int \frac{dy}{y} + C$.

$\Rightarrow \qquad \int \frac{dv}{\psi(v) - v} = \mathbf{log\ |y| + C, where\ } v = \frac{x}{y}$.

This equation is solved and v is replaced by $\frac{x}{y}$.

Remarks 1. The equation $\frac{dx}{dy} = \psi(x/y)$ can also be solved after interchanging x and y in the equation and again interchanging x and y in the solution of the given equation.

2. Sometimes, a given homogeneous differential equation is conveniently solved by expressing it in the form $\frac{dx}{dy} = \psi(x/y)$.

Example 42. *Solve:* $\frac{dx}{dy} = \frac{x}{y} + \sin\frac{x}{y}$.

Solution. We have $\quad \frac{dx}{dy} = \frac{x}{y} + \sin\frac{x}{y}$. ...(1)

This is a homogeneous differential equation of the form $\frac{dx}{dy} = \psi(x/y)$.

Let $\quad x = vy \quad \therefore \quad \frac{dx}{dy} = v + y\frac{dv}{dy}$

$\therefore$ (1) $\Rightarrow \quad v + y\frac{dv}{dy} = v + \sin v$

$\Rightarrow \qquad y\frac{dv}{dy} = \sin v \quad \Rightarrow \quad \text{cosec } v\, dv = \frac{dy}{y}$ (Variables are separate)

Integrating, we get $\quad \int \text{cosec } v\, dv = \int \frac{dy}{y} + \log C$.

$\Rightarrow \qquad \log\left|\tan\frac{v}{2}\right| = \log|y| + \log C$

NOTES

$$\Rightarrow \qquad \log\left|\tan\frac{x}{2y}\right| = \log C\,|y|$$

$$\Rightarrow \qquad \left|\tan\frac{x}{2y}\right| = C\,|y| \quad \Rightarrow \quad \tan\frac{x}{2y} = \pm Cy$$

$$\Rightarrow \qquad \mathbf{\tan\frac{x}{2y} = C_1 y}. \qquad \text{(Putting } C_1 = \pm C\text{)}$$

Example 43. *Solve:* $2ye^{x/y}\,dx + (y - 2xe^{x/y})\,dy = 0.$

Solution. We have $2ye^{x/y}\,dx + (y - 2xe^{x/y})\,dy = 0.$

$$\Rightarrow \qquad \frac{dx}{dy} = \frac{2xe^{x/y} - y}{2ye^{x/y}} \quad \Rightarrow \quad \frac{dx}{dy} = \frac{2(x/y)e^{x/y} - 1}{2e^{x/y}} \qquad \ldots(1)$$

This is a homogeneous differential equation of the form $\dfrac{dx}{dy} = \psi(x/y).$

Let $x = vy.$ $\quad\therefore\quad \dfrac{dx}{dy} = v + y\dfrac{dv}{dy}$

$$\therefore \quad (1) \quad \Rightarrow \qquad v + y\frac{dv}{dy} = \frac{2ve^v - 1}{2e^v}$$

$$\Rightarrow \qquad y\frac{dv}{dy} = \frac{2ve^v - 1}{2e^v} - v$$

$$\Rightarrow \qquad y\frac{dv}{dy} = -\frac{1}{2e^v} \quad \Rightarrow \quad 2e^v dv = -\frac{1}{y}dy$$

Integrating, we get $\displaystyle\int 2e^v\,dv = -\int \frac{1}{y}\,dy + C.$

$$\Rightarrow \qquad 2e^v = -\log|y| + C \quad \Rightarrow \quad \mathbf{2e^{x/y} + \log|y| = C}.$$

Example 44. *Solve:* $(1 + e^{x/y})\,dx + e^{x/y}\left(1 - \dfrac{x}{y}\right)dy = 0.$

Solution. We have $(1 + e^{x/y})\,dx + e^{x/y}\left(1 - \dfrac{x}{y}\right)dy = 0.$

$$\Rightarrow \qquad \frac{dx}{dy} = -\frac{e^{x/y}\left(1 - \dfrac{x}{y}\right)}{1 + e^{x/y}} \qquad \ldots(1)$$

This is a homogeneous differential equation of the form $\dfrac{dx}{dy} = \psi(x/y).$

Let $x = vy$ $\quad\therefore\quad \dfrac{dx}{dy} = v + y\,\dfrac{dv}{dy}$

$$\therefore \quad (1) \quad \Rightarrow \qquad v + y\frac{dv}{dy} = -\frac{e^v(1 - v)}{1 + e^v}$$

$$\Rightarrow \qquad y\,\frac{dv}{dy} = \frac{-e^v + ve^v}{1 + e^v} - v = \frac{-e^v + ve^v - v - ve^v}{1 + e^v}$$

$$\Rightarrow \qquad y\,\frac{dv}{dy} = \frac{-e^v - v}{1 + e^v}$$

$$\Rightarrow \quad \frac{1+e^v}{v+e^v}\,dv = -\frac{dy}{y} \qquad \text{(Variables are separate)}$$

$$\Rightarrow \quad \int \frac{1+e^v}{v+e^v}\,dv = -\int \frac{dy}{y} + \log C$$

$$\Rightarrow \quad \log|v+e^v| = -\log|y| + \log C$$

$$\Rightarrow \quad \log|v+e^v| = \log\frac{C}{|y|} \quad \Rightarrow \quad |y(v+e^v)| = C$$

$$\Rightarrow \quad y\left(\frac{x}{y}+e^{x/y}\right) = \pm C \quad \Rightarrow \quad \mathbf{x + ye^{x/y} = C_1}. \qquad (\text{Putting } C_1 = \pm C)$$

Example 45. *Solve:* $y\dfrac{dx}{dy}\sin\left(\dfrac{x}{y}\right) + y - x\sin\left(\dfrac{x}{y}\right) = 0,\ y(\pi/2) = 1.$

Solution. We have

$$y\frac{dx}{dy}\sin\left(\frac{x}{y}\right) + y - x\sin\left(\frac{x}{y}\right) = 0.$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{x\sin(x/y) - y}{y\sin(x/y)} \quad \Rightarrow \quad \frac{dx}{dy} = \frac{(x/y)\sin(x/y) - 1}{\sin(x/y)} \qquad ...(1)$$

This is a homogeneous differential equation of the form $\dfrac{dx}{dy} = \psi(x/y)$.

Let $x = vy$. $\quad \therefore \quad \dfrac{dx}{dy} = v + y\dfrac{dv}{dy}$

$$\therefore \quad (1) \Rightarrow \quad v + y\frac{dv}{dy} = \frac{v\sin v - 1}{\sin v} \quad \Rightarrow \quad y\frac{dv}{dy} = \frac{v\sin v - 1}{\sin v} - v$$

$$\Rightarrow \quad y\frac{dv}{dy} = -\frac{1}{\sin v} \quad \Rightarrow \quad \sin v\,dv = -\frac{dy}{y}$$

Integrating, we get $\displaystyle\int \sin v\,dv = -\int \frac{dy}{y} + C.$

$$\Rightarrow \quad -\cos v = -\log|y| + C$$

$$\Rightarrow \quad \log|y| = \cos(x/y) + C \qquad ...(2)$$

We have $y = 1$ when $x = \pi/2$.

$$\therefore \quad (2) \Rightarrow \quad \log|1| = \cos\left(\frac{\pi/2}{1}\right) + C$$

$$\Rightarrow \quad 0 = 0 + C \quad \Rightarrow \quad C = 0$$

$\therefore$ (2) $\Rightarrow$ $\mathbf{\log|y| = \cos(x/y)}$. This is the required solution.

EXERCISE I

Solve the following differential equations (Q. No. 1–25):

1. $(3xy + y^2)dx + (x^2 + xy)dy = 0$

2. $2xyy' = x^2 + 3y^2$

3. $(x^2 + xy)dy = (x^2 + y^2)dx$

4. $(x^2 - y^2)dx + 2xy\,dy = 0$

5. $x\dfrac{dy}{dx} + \dfrac{y^2}{x} = y$

6. $x^2y\,dx - (x^3 + y^3)dy = 0$

7. $x^2\dfrac{dy}{dx} = y(x + y)$

8. $y - x\dfrac{dy}{dx} = x + y\dfrac{dy}{dx}$

9. $x^2 \dfrac{dy}{dx} = \dfrac{y(x+y)}{2}$

10. $x \dfrac{dy}{dx} = y(\log y - \log x + 1)$

11. $\dfrac{dy}{dx} = \dfrac{y}{x} + \tan \dfrac{y}{x}$

12. $(x - y) \dfrac{dy}{dx} = x + 2y$

13. $x^2 y_1 = x^2 - 2y^2 + xy$

14. $(x^2 - y^2)dx + xy\, dy = 0$

15. $x^2\, dy + y(x + y)dx = 0$

16. $y\, dx + x\left(\log \dfrac{y}{x}\right)dy - 2x\, dy = 0$

17. $xy\left(\log \dfrac{y}{x}\right)dx + \left(y^2 - x^2 \log \dfrac{y}{x}\right)dy = 0$

18. $\dfrac{y}{x} \cos \dfrac{y}{x}\, dx - \left(\dfrac{x}{y} \sin \dfrac{y}{x} + \cos \dfrac{y}{x}\right)dy = 0$

19. $(y^2 - 2xy)dx = (x^2 - 2xy)dy$

20. $y^2\, dx + (x^2 - xy + y^2)dy = 0$

21. $2xy\, dx + (x^2 + 2y^2)dy = 0$

22. $(y^2 - x^2)dy = 3xy\, dx$

23. $x \cos\left(\dfrac{y}{x}\right)\dfrac{dy}{dx} = y \cos\left(\dfrac{y}{x}\right) + x$

24. $(x - y)dy - (x + y)dx = 0$

25. $x\dfrac{dy}{dx} - y + x \sin \dfrac{y}{x} = 0.$

Solve the following initial value problems (Q. No. 26–45):

26. $y^2 + x^2 \dfrac{dy}{dx} = xy \dfrac{dy}{dx},\ y(1) = 1$

27. $x(x^2 + 3y^2)dx + y(y^2 + 3x^2)dy = 0,\ y(1) = 1$

28. $(y^4 - 2x^3y)dx + (x^4 - 2xy^3)dy = 0,\ y(1) = 1$

29. $xe^{y/x} - y + x\dfrac{dy}{dx} = 0,\ y(e) = 0$

30. $(xe^{y/x} + y)dx = x\, dy,\ y(1) = 1$

31. $(x + y)dy + (x - y)dx = 0,\ y(1) = 1$

32. $2xy + y^2 - 2x^2 \dfrac{dy}{dx} = 0,\ y(1) = 2$

33. $2x^2 \dfrac{dy}{dx} - 2xy + y^2 = 0,\ y(e) = e$

34. $2ye^{x/y}\, dx + (y - 2xe^{x/y})dy = 0,\ y(0) = 1$

35. $(x^2 - y^2)dx + 2xy\, dy = 0,\ y(1) = 1$

36. $x^2 \dfrac{dy}{dx} = y^2 + 2xy,\ y(1) = 1$

37. $\dfrac{dy}{dx} - \dfrac{y}{x} + \operatorname{cosec} \dfrac{y}{x} = 0,\ y(1) = 0$

38. $(x\, dy - y\, dx)y \sin \dfrac{y}{x} = (y\, dx + x\, dy)x \cos \dfrac{y}{x},\ y(3) = \pi$

39. $x\dfrac{dy}{dx} - y + x \sin \dfrac{y}{x} = 0,\ y(2) = \pi.$

40. $x\dfrac{dy}{dx} \sin\left(\dfrac{y}{x}\right) + x - y \sin\left(\dfrac{y}{x}\right) = 0,\ y(1) = \dfrac{\pi}{2}$

41. $(3xy + y^2)\, dx + (x^2 + xy)dy = 0,\ y(1) = 1$

42. $x \cos\left(\dfrac{y}{x}\right)\dfrac{dy}{dx} = x + y \cos\left(\dfrac{y}{x}\right),\ y(1) = \dfrac{\pi}{4}$

43. $(x^2 + xy)dy = (x^2 + y^2)dx,\ y(1) = 0$

44. $(x - y)\dfrac{dy}{dx} = x + 2y,\ y(1) = 0$

45. $(x^2 + y^2)dy - xy\, dx = 0,\ y(0) = 1.$

Answers

1. $x^2y(2x + y) = C$

2. $x^2 + y^2 = Cx^3$

3. $\log |x| = \log (x - y)^2 + \dfrac{y}{x} + C$

4. $x = C(x^2 + y^2)$

5. $\log |x| = \dfrac{x}{y} + C$

6. $\dfrac{x^3}{3y^3} = \log |y| + C$

7. $y \log |x| + x + Cy = 0$

8. $\dfrac{1}{2} \log (x^2 + y^2) + \tan^{-1} \dfrac{y}{x} = C$

9. $xy^2 = C(y - x)^2$

10. $\log \dfrac{y}{x} = Cx$

NOTES

11. $x = C \sin \frac{y}{x}$

12. $\log |x^2 + xy + y^2| - 2\sqrt{3} \tan^{-1} \frac{x+2y}{\sqrt{3}\,x} = C$

13. $\frac{1}{2\sqrt{2}} \log \left| \frac{x+\sqrt{2}y}{x-\sqrt{2}y} \right| = \log |x| + C$

14. $y^2 = x^2(C - 2\log|x|)$

15. $x^2y = C(y + 2x)$

16. $Cy = \log \frac{y}{x} - 1$

17. $\log y^2 + \frac{x^2}{y^2}\left(\log \frac{y}{x} + \frac{1}{2}\right) = C$

18. $y \sin \frac{y}{x} = C$

19. $x^2y - xy^2 = C$

20. $y = Ce^{\tan^{-1}(y/x)}$

21. $3x^2y + 2y^3 = C$

22. $y^2(4x^2 - y^2)^3 = C$

23. $\sin \frac{y}{x} = \log |x| + C$

24. $\tan^{-1} \frac{y}{x} = \frac{1}{2} \log (x^2 + y^2) + C$

25. $1 - \cos \frac{y}{x} = \frac{C}{x} \sin \frac{y}{x}$

26. $\frac{y}{x} - \log |y| = 1$

27. $x^4 + 6x^2y^2 + y^4 = 8$

28. $x^3 + y^3 = 2xy$

29. $y = -x \log \log |x|$

30. $\log |x| = \frac{1}{e} - \frac{1}{e^{y/x}}$

31. $\log (x^2 + y^2) + 2\tan^{-1} \frac{y}{x} = \frac{\pi}{2} + \log 2$

32. $\frac{2x}{y} + \log |x| = 1$

33. $y \log ex = 2x$

34. $2e^{x/y} + \log |y| = 2$

35. $x^2 + y^2 = 2x$

36. $2y = x(x + y)$

37. $\cos \frac{y}{x} = 1 + \log |x|$

38. $2xy \cos \frac{y}{x} = 3\pi$

39. $x\left(\operatorname{cosec} \frac{y}{x} - \cot \frac{y}{x}\right) = 2$

40. $\log |x| = \cos \frac{y}{x}$

41. $x^2y^2 + 2x^3y = 3$

42. $\sin \frac{y}{x} = \log |x| + \frac{1}{\sqrt{2}}$

43. $\frac{y}{x} + 2\log |x - y| - \log |x| = 0$

44. $\frac{1}{2} \log |x^2 + y^2 + xy| + \frac{\sqrt{3}\pi}{6} = \sqrt{3} \tan^{-1}\left(\frac{x+2y}{\sqrt{3}x}\right)$

45. $x^2 = 2y^2 \log y$.

Solution of $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$, where $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ by Reducing it to a Homogeneous Equation

Consider the differential equation

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}, \quad \text{where } \frac{a_1}{a_2} \neq \frac{b_1}{b_2}. \qquad \ldots(1)$$

We substitute $x = X + h$ and $y = Y + k$, where h and k are constants to be properly chosen.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dY} \times \frac{dY}{dX} \times \frac{dX}{dx} = 1 \times \frac{dY}{dX} \times 1 = \frac{dY}{dX}$$

NOTES

$\therefore$ (1) $\Rightarrow$ $$\frac{dY}{dX} = \frac{a_1(X+h)+b_1(Y+k)+c_1}{a_2(X+h)+b_2(Y+k)+c_2}$$

$\Rightarrow$ $$\frac{dY}{dX} = \frac{a_1X+b_1Y+(a_1h+b_1k+c_1)}{a_2X+b_2Y+(a_2h+b_2k+c_2)} \qquad ...(2)$$

The constants h and k are chosen so that $a_1h + b_1k + c_1 = 0$ and $a_2h + b_2k + c_2 = 0$.

$\therefore$ (2) $\Rightarrow$ $$\frac{dY}{dX} = \frac{a_1X+b_1Y}{a_2X+b_2Y} \qquad ...(3)$$

This is a homogeneous differential equation and can be solved by putting $Y = VX$.

$Y = VX$ $\Rightarrow$ $$\frac{dY}{dX} = V + X\frac{dV}{dX}$$

$\therefore$ (3) $\Rightarrow$ $$V + X\frac{dV}{dX} = \frac{a_1X+b_1VX}{a_2X+b_2VX} = \frac{a_1+b_1V}{a_2+b_2V}$$

$\Rightarrow$ $$X\frac{dV}{dX} = \frac{a_1+b_1V}{a_2+b_2V} - V = \frac{a_1+b_1V-a_2V-b_2V^2}{a_2+b_2V}$$

$\Rightarrow$ $$\frac{a_2+b_2V}{a_1+(b_1-a_2)V-b_2V^2}\,dV = \frac{dX}{X} \qquad ...(4)$$

In the differential equation (4), the variables X and V are separated.

Integrating (4), we get $\displaystyle\int \frac{a_2+b_2V}{a_1+(b_1-a_2)V-b_2V^2}\,dV = \int \frac{dX}{X} + C.$

$\Rightarrow$ $$\boldsymbol{\int \frac{a_2+b_2V}{a_1+(b_1-a_2)V-b_2V^2}\,dV = \log|X| + C,}$$

where $V = Y/X$, $X = x - h$ and $Y = y - k$.

This represents the general solution of the differential equation (1).

Remark. If $\frac{a_1}{a_2} = \frac{b_1}{b_2}$ in the differential equation $\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ then it can be easily solved by putting $z = a_1x + b_1y$ or $z = a_2x + b_2y$.

Working Steps for Solving $\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$, where $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

Step I. Put $x = X + h$ and $y = Y + k$. Given differential equation reduces to

$$\frac{dY}{dX} = \frac{a_1X+b_1Y+(a_1h+b_1k+c_1)}{a_2X+b_2Y+(a_2h+b_2k+c_2)}.$$

Step II. Solve $a_1h + b_1k + c_1 = 0$ and $a_2h + b_2k + c_2 = 0$ to get the values of h and k. The resultant equation $\frac{dY}{dX} = \frac{a_1X+b_1Y}{a_2X+b_2Y}$ is a homogeneous differential equation.

Step III. Put $Y = VX$. This gives a differential equation in X and V with variables separated.

Step IV. Solve this differential equation and put $V = Y/X$, $X = x - h$ and $Y = y - k$ to get the answer in original variables x and y.

NOTES

SOLVED EXAMPLES

Example 46. *Solve:* $\dfrac{dy}{dx} = \dfrac{x+2y-5}{2x+y-4}$.

Solution. We have $\dfrac{dy}{dx} = \dfrac{x+2y-5}{2x+y-4}$. ...(1)

Here $\dfrac{a_1}{a_2} = \dfrac{1}{2}$ and $\dfrac{b_1}{b_2} = \dfrac{2}{1} = 2 \quad \therefore \quad \dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$

Let $x = X + h$ and $y = Y + k$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dY} \times \frac{dY}{dX} \times \frac{dX}{dx} = 1 \times \frac{dY}{dX} \times 1 = \frac{dY}{dX}$$

$$\therefore \quad (1) \Rightarrow \quad \frac{dY}{dX} = \frac{(X+h)+2(Y+k)-5}{2(X+h)+(Y+k)-4}$$

$$\Rightarrow \quad \frac{dY}{dX} = \frac{X+2Y+(h+2k-5)}{2X+Y+(2h+k-4)} \qquad ...(2)$$

Let h and k be such that $h + 2k - 5 = 0$ and $2h + k - 4 = 0$.

$$\therefore \quad h = 1, k = 2 \qquad \text{(On simplification)}$$

$$\therefore \quad (2) \Rightarrow \quad \frac{dY}{dX} = \frac{X+2Y}{2X+Y} \qquad ...(3)$$

This is a homogeneous differential equation.

Let $Y = VX$. $\therefore \quad \dfrac{dY}{dX} = V + X\dfrac{dV}{dX}$

$$\therefore \quad (3) \Rightarrow \quad V + X\frac{dV}{dX} = \frac{X+2(VX)}{2X+VX} = \frac{1+2V}{2+V}$$

$$\Rightarrow \quad X\frac{dV}{dX} = \frac{1+2V}{2+V} - V = \frac{1+2V-2V-V^2}{2+V} = \frac{1-V^2}{2+V}$$

$$\Rightarrow \quad \frac{2+V}{1-V^2}\,dV = \frac{dX}{X} \quad \Rightarrow \quad \int \frac{2+V}{1-V^2}\,dV = \int \frac{dX}{X} + \log C$$

(Variables are separate)

$$\Rightarrow \quad \int \frac{2+V}{(1+V)(1-V)}\,dV = \log |X| + \log C$$

$$\Rightarrow \quad \int \left[\frac{1}{(1+V)(2)} + \frac{3}{2(1-V)}\right] dV = \log C|X|$$

$$\Rightarrow \quad \frac{1}{2}\log|1+V| + \frac{3}{2}\cdot\frac{\log|1-V|}{-1} = \log C|X|$$

$$\Rightarrow \quad \log\left|\frac{1+V}{(1-V)^3}\right| = \log C^2X^2$$

$$\Rightarrow \quad \left|\frac{1+Y/X}{(1-Y/X)^3}\right| = C^2X^2 \quad \Rightarrow \quad \frac{X+Y}{(X-Y)^3} = \pm C^2$$

$$\Rightarrow \quad X + Y = C_1(X-Y)^3, \quad \text{where } C_1 = \pm C^2$$

$$\Rightarrow \quad (x-1) + (y-2) = C_1((x-1)-(y-2))^3$$

$$\Rightarrow \quad \mathbf{x + y - 3 = C_1(x - y + 1)^3}.$$

NOTES

EXERCISE J

Solve the following differential equations:

1. $\dfrac{dy}{dx} = \dfrac{x+2y-3}{2x+y+3}$ **2.** $\dfrac{dy}{dx} = \dfrac{y-x+1}{y+x-5}$

3. $\dfrac{dy}{dx} = \dfrac{x-y+1}{x+y-2}$ **4.** $\dfrac{dy}{dx} = \dfrac{2x-y+1}{x+2y-3}$.

Answers

1. $x + y = C(x - y + 6)^3$ **2.** $\tan^{-1}\dfrac{y-2}{x-3} + \dfrac{1}{2}\log(x^2 + y^2 - 6x - 4y + 13) = C$

3. $y^2 + 2xy - x^2 - 2x - 4y = C$ **4.** $x^2 - y^2 - xy + x + 3y = C$.

SOLUTION OF LINEAR DIFFERENTIAL EQUATION $\dfrac{dy}{dx}$ + Py = Q, WHERE P AND Q ARE FUNCTIONS OF x OR CONSTANTS

Let $$\frac{dy}{dx} + Py = Q \qquad ...(1)$$

be a linear differential equation, where P and Q are functions of x or constants.

Multiplying both sides of (1) by $e^{\int P\,dx}$, we get

$$e^{\int P\,dx}\frac{dy}{dx} + e^{\int P\,dx}\,Py = Qe^{\int P\,dx}.$$

$$\Rightarrow \quad e^{\int P\,dx}\frac{dy}{dx} + \frac{d}{dx}(e^{\int P\,dx}).\,y = Qe^{\int P\,dx} \qquad \left(\because \frac{d}{dx}\int P dx = P\right)$$

$$\Rightarrow \quad \frac{d}{dx}(ye^{\int P\,dx}) = Qe^{\int P\,dx}$$

$$\Rightarrow \quad \int\left[\frac{d}{dx}(ye^{\int P\,dx})\right]dx = \int Qe^{\int P\,dx}\,dx + C$$

$$\Rightarrow \quad \boldsymbol{ye^{\int P\,dx} = \int Qe^{\int P\,dx}\,dx + C}.$$

This is the general solution of linear differential equation (1). The function $e^{\int P\,dx}$ is called the **integrating factor** (I.F.) of (1).

Thus, the solution of (1) can also be written as

$$\boldsymbol{y(\text{I.F.}) = \int Q(\text{I.F.})dx + C}.$$

Remark. In evaluating integrating factor (I.F.), the results $e^{\log f(x)} = f(x)$ is frequently used.

NOTES

Working Steps for Solving $\frac{dy}{dx} + Py = Q$

Step I. If the coefficient of $\frac{dy}{dx}$ is not unity, it must be made unity by dividing the equation by the coefficient of $\frac{dy}{dx}$.

Step II. Identify P and Q and make sure that these are functions of x or constants.

Step III. Evaluate $\int$ P dx.

Step IV. Find $e^{\int P\,dx}$. This is the integrating factor (I.F.).

Step V. Put the value of I.F. in the general solution y(I.F.) = $\int$ Q(I.F.) dx + C and simplify it. This gives the general solution of the given differential equation.

SOLVED EXAMPLES

Example 47. *Solve:* $\frac{dy}{dx} + 2y = e^{-x}$.

Solution. We have $\frac{dy}{dx} + 2y = e^{-x}$. ...(1)

This is a linear differential equation. Here P = 2 and Q = e^{-x}.

$$\int P\,dx = \int 2\,dx = 2x \qquad \therefore \quad \text{I.F.} = e^{\int P\,dx} = e^{2x}$$

The solution of (1) is $y(\text{I.F.}) = \int Q(\text{I.F.})\,dx + C$.

$$\Rightarrow \quad ye^{2x} = \int e^{-x}\,e^{2x}\,dx + C \quad \Rightarrow \quad ye^{2x} = \int e^{x}\,dx + C$$

$$\Rightarrow \quad ye^{2x} = e^{x} + C \quad \Rightarrow \quad \mathbf{y = e^{-x} + Ce^{-2x}}.$$

Example 48. *Solve:* $y' - 2y = \cos 3x$.

Solution. We have $\frac{dy}{dx} - 2y = \cos 3x$. ...(1)

This is a linear differential equation. Here P = –2 and Q = cos 3x.

$$\int P\,dx = \int -2\,dx = -2x \quad \therefore \quad \text{I.F.} = e^{\int P\,dx} = e^{-2x}$$

The solution of (1) is $y(\text{I.F.}) = \int Q(\text{I.F.})\,dx + C$.

$$\Rightarrow \quad ye^{-2x} = \int \cos 3x\,.\,e^{-2x}\,dx + C \quad \Rightarrow \quad ye^{-2x} = \int e^{-2x}\cos 3x\,dx + C$$

$$\Rightarrow \quad y = e^{2x}\int e^{-2x}\cos 3x\,dx + Ce^{2x} \qquad \text{...(2)}$$

Let $\quad I = \int e^{-2x}\cos 3x\,dx$

$$\therefore \quad I = e^{-2x}\,\frac{\sin 3x}{3} - \int -2\,e^{-2x}\,\frac{\sin 3x}{3}\,dx = \frac{e^{-2x}\sin 3x}{3} + \frac{2}{3}\int e^{-2x}\sin 3x\,dx$$

NOTES

$$= \frac{e^{-2x}\sin 3x}{3} + \frac{2}{3}\left[e^{-2x}\left(\frac{-\cos 3x}{3}\right) - \int -2e^{-2x}\left(\frac{-\cos 3x}{3}\right)dx\right]$$

$$= \frac{e^{-2x}\sin 3x}{3} - \frac{2}{9}e^{-2x}\cos 3x - \frac{4}{9}\int e^{-2x}\cos 3x\,dx$$

$$\therefore \quad I = \frac{e^{-2x}}{9}(3\sin 3x - 2\cos 3x) - \frac{4}{9}I$$

$$\Rightarrow \quad \left(1+\frac{4}{9}\right)I = \frac{e^{-2x}}{9}(3\sin 3x - 2\cos 3x) \quad \Rightarrow \quad I = \frac{e^{-2x}}{13}(3\sin 3x - 2\cos 3x)$$

$$\therefore \quad (2) \Rightarrow \quad y = e^{2x}\left[\frac{e^{-2x}}{13}(3\sin 3x - 2\cos 3x)\right] + Ce^{2x}$$

$$\therefore \quad \mathbf{y = \frac{1}{13}(3\sin 3x - 2\cos 3x) + Ce^{2x}}.$$

Example 49. *Solve:* $\frac{dy}{dx} + ay = e^{mx}$.

Solution. We have $\frac{dy}{dx} + ay = e^{mx}$. ...(1)

This is a linear differential equation. Here $P = a$ and $Q = e^{mx}$.

$$\int P\,dx = \int a\,dx = ax \quad \therefore \quad \text{I.F.} = e^{\int P\,dx} = e^{ax}$$

The solution of (1) is $y(\text{I.F.}) = \int Q(\text{I.F.})\,dx + C$.

$$\Rightarrow \quad ye^{ax} = \int e^{mx}e^{ax}\,dx + C \quad \Rightarrow \quad ye^{ax} = \int e^{(a+m)x}\,dx + C$$

$$\Rightarrow \quad ye^{ax} = \frac{e^{(a+m)x}}{a+m} + C \quad \Rightarrow \quad y = \frac{e^{-ax}\cdot e^{(a+m)x}}{a+m} + C\cdot e^{-ax}$$

(Provided $a + m \neq 0$)

$$\Rightarrow \quad \mathbf{y = \frac{e^{mx}}{a+m} + Ce^{-ax}}.$$

This is the required solution of the given differential equation.

If $a + m = 0$, then the given differential equation (1) becomes $\frac{dy}{dx} - my = e^{mx}$.

Here $P = -m$ and $Q = e^{mx}$.

$$\int P\,dx = \int -m\,dx = -mx \quad \therefore \quad \text{I.F.} = e^{\int P\,dx} = e^{-mx}$$

$\therefore$ The solution is $ye^{-mx} = \int e^{-mx}e^{mx}\,dx + C$.

$$\Rightarrow \quad y = e^{mx}\left[\int 1\cdot dx + C\right] \quad \Rightarrow \mathbf{y = e^{mx}(x + C)}.$$

This is the required solution of the given differential equation provided $a + m = 0$.

Example 50. *Solve:* $\frac{dy}{dx} - \frac{y}{x} = 2x^2$.

Solution. We have $\frac{dy}{dx} - \frac{y}{x} = 2x^2$. ...(1)

NOTES

This is a linear differential equation. Here $P = -\frac{1}{x}$ and $Q = 2x^2$.

$$\int P\,dx = \int -\frac{1}{x}\,dx = -\log|x| = \log\frac{1}{x} \qquad \text{(Assuming } x > 0\text{)}$$

$$\therefore \qquad \text{I.F.} = e^{\int P\,dx} = e^{\log\frac{1}{x}} = \frac{1}{x}$$

The solution of (1) is

$$y(\text{I.F.}) = \int Q(\text{I.F.})\,dx + C. \quad \therefore \quad y\left(\frac{1}{x}\right) = \int 2x^2\left(\frac{1}{x}\right)dx + C$$

$$\Rightarrow \qquad \frac{y}{x} = 2\int x\,dx + C \quad \Rightarrow \quad \frac{y}{x} = 2\cdot\frac{x^2}{2} + C \quad \Rightarrow \quad \mathbf{y = x^3 + Cx}.$$

Example 51. *Solve:* $\cos^2 x\,\frac{dy}{dx} + y = \tan x$.

Solution. We have $\cos^2 x\,\frac{dy}{dx} + y = \tan x$. $\therefore$ $\frac{dy}{dx} + y\sec^2 x = \tan x\sec^2 x$ ...(1)

This is a linear differential equation. Here $P = \sec^2 x$ and $Q = \tan x\sec^2 x$.

$$\int P\,dx = \int \sec^2 x\,dx = \tan x \quad \therefore \quad \text{I.F.} = e^{\int P\,dx} = e^{\tan x}$$

The solution of (1) is $y(\text{I.F.}) = \int Q(\text{I.F.})\,dx + C$.

$$\therefore \qquad ye^{\tan x} = \int \tan x\sec^2 x\,e^{\tan x}\,dx + C \qquad \text{...(2)}$$

$$\text{Let} \qquad I = \int \tan x\sec^2 x\,e^{\tan x}\,dx. \qquad z = \tan x \;\Rightarrow\; dz = \sec^2 x\,dx$$

$$\therefore \qquad I = \int ze^z\,dz = ze^z - \int 1.e^z\,dz$$

$$= ze^z - e^z = (z-1)\,e^z = (\tan x - 1)\,e^{\tan x}$$

$$\therefore \quad (2) \quad \Rightarrow \qquad y\,e^{\tan x} = (\tan x - 1)e^{\tan x} + C$$

$$\Rightarrow \qquad \mathbf{y = \tan x - 1 + Ce^{-\tan x}}.$$

Example 52. *Solve:* $(x^2 - 1)\,\frac{dy}{dx} + 2(x + 2)y = 2(x + 1)$.

Solution. We have $(x^2 - 1)\,\frac{dy}{dx} + 2(x + 2)y = 2(x + 1)$.

$$\therefore \qquad \frac{dy}{dx} + \frac{2(x+2)}{x^2-1}\,y = \frac{2}{x-1} \qquad \text{...(1)}$$

This is a linear differential equation. Here $P = \frac{2(x+2)}{x^2-1}$ and $Q = \frac{2}{x-1}$.

$$\int P\,dx = \int \frac{2(x+2)}{x^2-1}\,dx = \int\left(\frac{3}{x-1} - \frac{1}{x+1}\right)dx$$

$$= 3\log(x-1) - \log(x+1) = \log\frac{(x-1)^3}{x+1}$$

(Assuming $x - 1, x + 1 > 0$)

$$\text{I.F.} = e^{\int P\,dx} = e^{\log\frac{(x-1)^3}{x+1}} = \frac{(x-1)^3}{x+1}$$

The solution of (1) is $y(\text{I.F.}) = \int Q(\text{I.F.})\,dx + C$.

NOTES

$$\therefore \quad y\frac{(x-1)^3}{x+1} = \int \frac{2}{x-1} \cdot \frac{(x-1)^3}{x+1}\, dx + C \;= 2\int \frac{x^2-2x+1}{x+1}\, dx + C$$

$$= 2\int \left(x - 3 + \frac{4}{x+1}\right) dx + C = 2\left[\frac{x^2}{2} - 3x + 4\log(x+1)\right] + C$$

$$\therefore \quad \boldsymbol{y\frac{(x-1)^3}{x+1} = x^2 - 6x + 8\log(x+1) + C}.$$

Example 53. *Solve:* $\sin x \dfrac{dy}{dx} + 3y = \cos x.$

Solution. We have $\sin x \dfrac{dy}{dx} + 3y = \cos x.$ $\therefore$ $\dfrac{dy}{dx} + 3y \operatorname{cosec} x = \cot x$ ...(1)

This is a linear differential equation. Here P = 3cosec x and Q = cot x.

$$\int P\, dx = \int 3 \operatorname{cosec} x\, dx = 3\log\tan\frac{x}{2} = \log\tan^3\frac{x}{2}$$

$$\therefore \quad \text{I.F.} = e^{\int P\,dx} = e^{\log\tan^3\frac{x}{2}} = \tan^3\frac{x}{2}$$

The solution of (1) is $y(\text{I.F.}) = \int Q(\text{I.F.})\, dx + C.$

$$\Rightarrow \quad y\tan^3\frac{x}{2} = \int \cot x \tan^3\frac{x}{2}\, dx + C = \int \frac{1-\tan^2\frac{x}{2}}{2\tan\frac{x}{2}} \cdot \tan^3\frac{x}{2}\, dx + C$$

$$= \frac{1}{2}\int \left(\tan^2\frac{x}{2} - \tan^4\frac{x}{2}\right) dx + C$$

$$= \frac{1}{2}\int \left[\tan^2\frac{x}{2} - \tan^2\frac{x}{2}\left(\sec^2\frac{x}{2} - 1\right)\right] dx + C$$

$$= \frac{1}{2}\int \left[2\tan^2\frac{x}{2} - \tan^2\frac{x}{2}\sec^2\frac{x}{2}\right] dx + C$$

$$= \frac{1}{2}\int \left[2\left(\sec^2\frac{x}{2} - 1\right) - \tan^2\frac{x}{2}\sec^2\frac{x}{2}\right] dx + C$$

$$= \frac{1}{2}\left[4\tan\frac{x}{2} - 2x - \frac{2}{3}\tan^3\frac{x}{2}\right] + C = 2\tan\frac{x}{2} - x - \frac{1}{3}\tan^3\frac{x}{2} + C$$

$$\therefore \quad y\tan^3\frac{x}{2} = 2\tan\frac{x}{2} - x - \frac{1}{3}\tan^3\frac{x}{2} + C$$

$$\Rightarrow \quad \boldsymbol{\left(y + \frac{1}{3}\right)\tan^3\frac{x}{2} = 2\tan\frac{x}{2} - x + C}.$$

Example 54. *Solve:* $\left(\dfrac{e^{-2\sqrt{x}}}{\sqrt{x}} - \dfrac{y}{\sqrt{x}}\right)\dfrac{dx}{dy} = 1 \quad (x \neq 0).$

Solution. We have $\left(\dfrac{e^{-2\sqrt{x}}}{\sqrt{x}} - \dfrac{y}{\sqrt{x}}\right)\dfrac{dx}{dy} = 1$ *i.e.,* $\dfrac{dy}{dx} = \dfrac{e^{-2\sqrt{x}}}{\sqrt{x}} - \dfrac{y}{\sqrt{x}}$

$$\Rightarrow \quad \frac{dy}{dx} + \left(\frac{1}{\sqrt{x}}\right)y = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \qquad \text{...(1)}$$

This is a linear differential equation. Here $P = \dfrac{1}{\sqrt{x}}$ and $Q = \dfrac{e^{-2\sqrt{x}}}{\sqrt{x}}$.

$$\int P\,dx = \int \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x} \quad \therefore \quad \text{I.F.} = e^{\int P\,dx} = e^{2\sqrt{x}}$$

The solution of (1) is $y(\text{I.F.}) = \int Q(\text{I.F.})\,dx + C$.

$$\Rightarrow \qquad ye^{2\sqrt{x}} = \int \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \cdot e^{2\sqrt{x}}\,dx + C = \int x^{-1/2}\,dx + C = 2\sqrt{x} + C$$

$$\therefore \qquad \boldsymbol{ye^{2\sqrt{x}} = 2\sqrt{x} + C}.$$

Example 55. *Solve:* $x\dfrac{dy}{dx} + y - x + xy\cot x = 0 \quad (x \neq 0)$.

Solution. We have $x\dfrac{dy}{dx} + y - x + xy\cot x = 0$ *i.e.,* $\dfrac{dy}{dx} + \left(\dfrac{1}{x} + \cot x\right)y = 1$. ...(1)

This is a linear differential equation. Here $P = \dfrac{1}{x} + \cot x$ and $Q = 1$.

$$\int P\,dx = \int \left(\frac{1}{x} + \cot x\right)dx = \log|x| + \log|\sin x|$$

$$= \log|x\sin x| = \log(x\sin x) \qquad \text{(Assuming } x\sin x > 0)$$

$$\therefore \qquad \text{I.F.} = e^{\int P\,dx} = e^{\log(x\sin x)} = x\sin x$$

The solution of (1) is $y(\text{I.F.}) = \int Q(\text{I.F.})\,dx + C$.

$$\Rightarrow \qquad y(x\sin x) = \int 1\cdot(x\sin x)\,dx + C$$

$$= x(-\cos x) - \int 1\cdot(-\cos x)\,dx + C = -x\cos x + \sin x + C$$

$$\Rightarrow \qquad \boldsymbol{y = -\cot x + \frac{1}{x} + \frac{C}{x\sin x}}.$$

Example 56. *Solve:* $\dfrac{dy}{dx} - 3y\cot x = \sin 2x,\ y = 2$ *when* $x = \dfrac{\pi}{2}$.

Solution. We have $\dfrac{dy}{dx} - (3\cot x)y = \sin 2x$. ...(1)

This is a linear differential equation. Here $P = -3\cot x$ and $Q = \sin 2x$.

$$\int P\,dx = \int -3\cot x\,dx = -3\log|\sin x| = -3\log\sin x = \log(\sin x)^{-3}$$

(Assuming $\sin x > 0$)

$$\therefore \qquad \text{I.F.} = e^{\int P\,dx} = e^{\log(\sin x)^{-3}} = (\sin x)^{-3}$$

The solution of (1) is

$$y(\text{I.F.}) = \int Q(\text{I.F.})\,dx + C.$$

$$\Rightarrow \qquad y(\sin x)^{-3} = \int \sin 2x(\sin x)^{-3}\,dx + C$$

$$= 2\int (\sin x)^{-2}\cos x\,dx + C = 2\frac{(\sin x)^{-1}}{-1} + C$$

NOTES

$\therefore \qquad y = -2\sin^2 x + C\sin^3 x \qquad ...(2)$

Now, $y = 2$ when $x = \pi/2$.

$\therefore$ (2) $\Rightarrow$ $2 = -2\sin^2 \dfrac{\pi}{2} + C\sin^3 \dfrac{\pi}{2} \Rightarrow 2 = -2 + C(1) \Rightarrow C = 4$

$\therefore$ (2) $\Rightarrow$ $\mathbf{y = -2\sin^2 x + 4\sin^3 x}$.

Example 57. *Solve:* $\tan x \dfrac{dy}{dx} = 2x\tan x + x^2 - y$, *$y = 0$ when* $x = \dfrac{\pi}{2}$.

Solution. We have

$$\tan x \frac{dy}{dx} = 2x\tan x + x^2 - y. \qquad ...(1)$$

$$\Rightarrow \qquad \frac{dy}{dx} = 2x + \frac{x^2}{\tan x} - \frac{y}{\tan x}$$

$$\Rightarrow \qquad \frac{dy}{dx} + (\cot x)\,y = 2x + x^2\cot x.$$

This is a linear differential equation. Here $P = \cot x$, and $Q = 2x + x^2\cot x$.

$$\int P\,dx = \int \cot x\,dx = \log|\sin x| = \log \sin x \qquad \text{(Assuming } \sin x > 0)$$

$$\therefore \qquad \text{I.F.} = e^{\int P dx} = e^{\log \sin x} = \sin x$$

The solution of (1) is

$$y(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + C.$$

$$\Rightarrow \qquad y\sin x = \int (2x + x^2\cot x)\sin x\,dx + C$$

$$\Rightarrow \qquad y\sin x = \int 2x\sin x\,dx + \int x^2\cos x\,dx + C$$

$$= \int 2x\sin x\,dx + \left[x^2\sin x - \int 2x\sin x\,dx\right] + C$$

$$= x^2\sin x + C$$

$$\Rightarrow \qquad y = x^2 + C\,\text{cosec}\,x \qquad ...(2)$$

Now, $y = 0$ when $x = \dfrac{\pi}{2}$.

$$\Rightarrow \qquad 0 = \left(\frac{\pi}{2}\right)^2 + C\,\text{cosec}\,\frac{\pi}{2}$$

$$\Rightarrow \qquad 0 = \frac{\pi^2}{4} + C\,.\,(1) \quad \Rightarrow \quad C = -\frac{\pi^2}{4}$$

$$\therefore \quad (2) \Rightarrow \qquad \mathbf{y = x^2 - \frac{\pi^2}{4}\,\text{cosec}\,x}.$$

EXERCISE K

Solve the following differential equations (Q. No. 1–2):

1. (*i*) $\dfrac{dy}{dx} + 2y = e^{4x}$ (*ii*) $\dfrac{dy}{dx} - 2y = e^{3x}$

NOTES

(iii) $\frac{dy}{dx} + 2y = 6e^x$ (iv) $4\frac{dy}{dx} + 8y = 5e^{-3x}$

2. (i) $\frac{dy}{dx} + y = 1$ (ii) $\frac{dy}{dx} + y = e^x$

(iii) $\frac{dy}{dx} + y = e^{-3x}$ (iv) $\frac{dy}{dx} - 4y = e^x$.

Solve the following differential equations (Q. No. 3–15):

3. (i) $\frac{dy}{dx} + y = 2 - x$ (ii) $(y + 3x^2)\frac{dx}{dy} = x$

(iii) $x\,dy + (y - x^3)dx = 0$ (iv) $x\,dy - (y + 2x^2)dx = 0$

4. (i) $\frac{dy}{dx} + 2y = \sin x$ (ii) $\frac{dy}{dx} - y = \cos x$

(iii) $\frac{dy}{dx} + 2y = xe^{4x}$ (iv) $\frac{dy}{dx} + y = \cos 2x$

5. (i) $\frac{dy}{dx} + y = \frac{1 + x\log x}{x}$ (ii) $\frac{dy}{dx} + y = \frac{1 + \sin x}{1 + \cos x}$

(iii) $\frac{dy}{dx} + y = \cos x - \sin x$ (iv) $x\frac{dy}{dx} + 2y = x\cos x$

6. (i) $\frac{dy}{dx} - \frac{y}{x} = 2x^2$ (ii) $2x\frac{dy}{dx} + y = 6x^3$

(iii) $\frac{dy}{dx} + \frac{y}{x} = x^2$ (iv) $\frac{dy}{dx} + \frac{y}{2x} = 3x^2$

7. (i) $\sec x\frac{dy}{dx} - y = \sin x$ (ii) $\frac{dy}{dx} + \frac{y}{x} = e^x$

(iii) $x\frac{dy}{dx} + 2y = x^2\log x$ (iv) $\frac{dy}{dx} + \frac{y}{x} = \cos x + \frac{\sin x}{x}$

8. (i) $\frac{dy}{dx} = y\tan x - 2\sin x$ (ii) $\frac{dy}{dx} + y\sec x = \tan x$

(iii) $\frac{dy}{dx} - y\tan x = e^x$ (iv) $\cos^3 x\frac{dy}{dx} + y\cos x = \sin x$

9. (i) $x\frac{dy}{dx} - y = x + 1$ (ii) $(1 + x^2)dy + 2xy\,dx = \cot x\,dx$

(iii) $x\frac{dy}{dx} + 2y = x^2$ (iv) $(1 + x^2)\frac{dy}{dx} + y = \tan^{-1} x$

10. (i) $x\log x\frac{dy}{dx} + y = 2\log x$ (ii) $\sqrt{x}\frac{dy}{dx} + y = e^{-2\sqrt{x}}$

(iii) $\frac{dy}{dx} + \frac{3x^2}{1 + x^3}y = \frac{\sin^2 x}{1 + x^3}$ (iv) $(1 + x^2)\frac{dy}{dx} + y = e^{\tan^{-1} x}$

11. (i) $\frac{dy}{dx} + \frac{4x}{x^2 + 1}y = \frac{1}{(x^2 + 1)^3}$ (ii) $\frac{dy}{dx} + \frac{4x}{x^2 + 1}y = -\frac{1}{(x^2 + 1)^2}$

(iii) $\frac{dy}{dx} + \frac{y}{x\log x} = \frac{1}{x}$ (iv) $\frac{dy}{dx} + y\tan x = 2x + x^2\tan x$

12. (i) $x\frac{dy}{dx} - ay = x + 1$ (ii) $\frac{dy}{dx} - y\tan x = 2\sin x$

(iii) $\frac{dy}{dx} + 2y\tan x = \sin x$ (iv) $\frac{dy}{dx} + y\cot x = 2\cos x$

NOTES

13. (*i*) $x \log x \dfrac{dy}{dx} + y = \dfrac{2}{x} \log x$ (*ii*) $(x^2 - 1) \dfrac{dy}{dx} + 2xy = \dfrac{2}{x^2 - 1}$

14. (*i*) $(x^2 + 1) \dfrac{dy}{dx} + 2xy = \sqrt{x^2 + 4}$ (*ii*) $x \log x \dfrac{dy}{dx} + y = \log x$

(*iii*) $\dfrac{dy}{dx} + y \tan x = x^2 \cos^2 x$ (*iv*) $\dfrac{dy}{dx} + \dfrac{x + y \cos x}{1 + \sin x} = 0$

15. (*i*) $\sin x \dfrac{dy}{dx} + y \cos x = \cos x \sin^2 x$ (*ii*) $\dfrac{dy}{dx} + y \cot x = x^2 \cot x + 2x$

(*iii*) $(1 + x^2) \dfrac{dy}{dx} - 2xy = (x^2 + 2)(x^2 + 1)$

(*iv*) $(1 - x^2) \dfrac{dy}{dx} + xy = ax.$

Solve the following initial value problems (Q.No. 16–18):

16. (*i*) $x \dfrac{dy}{dx} - y = (x + 1)e^{-x}, y(1) = 0$ (*ii*) $\dfrac{dy}{dx} + y \cot x = 4x \operatorname{cosec} x, y(\pi/2) = 0$

(*iii*) $x \dfrac{dy}{dx} + y = x \cos x + \sin x, y(\pi/2) = 1$

(*iv*) $(x^2 + 1) \dfrac{dy}{dx} - 2xy = (x^4 + 2x^2 + 1) \cos x, y(0) = 0$

17. (*i*) $\dfrac{dy}{dx} + 2y \tan x = \sin x, y(\pi/3) = 0$ (*ii*) $\dfrac{dy}{dx} + y \sec^2 x = \tan x \sec^2 x, y(0) = 1$

(*iii*) $\dfrac{dy}{dx} - y = \cos x, y(0) = 1$ (*iv*) $x\dfrac{dy}{dx} + 2y = x^2, y(2) = 1$

18. (*i*) $(1 + x^2) \dfrac{dy}{dx} + 2xy = \dfrac{1}{1 + x^2}, y(1) = 0$ (*ii*) $\dfrac{dy}{dx} + y \cot x = 2x + x^2 \cot x, x \neq 0, y(\pi/2) = 0$

(*iii*) $(1 + y + x^2y)dx + (x + x^3)dy = 0, y(1) = 0$

(*iv*) $\cos x\, dy = \sin x(\cos x - 2y)dx, y(\pi/3) = 0.$

Answers

1. (*i*) $y = \dfrac{e^{4x}}{6} + Ce^{-2x}$ (*ii*) $y = e^{3x} + Ce^{2x}$

(*iii*) $y = 2e^x + Ce^{-2x}$ (*iv*) $y = -\dfrac{5}{4} e^{-3x} + Ce^{-2x}$

2. (*i*) $y = 1 + Ce^{-x}$ (*ii*) $y = \dfrac{1}{2} e^x + Ce^{-x}$

(*iii*) $y = -\dfrac{1}{2} e^{-3x} + Ce^{-x}$ (*iv*) $y = -\dfrac{1}{3} e^x + Ce^{4x}$

3. (*i*) $y = 3 - x + Ce^{-x}$ (*ii*) $y = 3x^2 + Cx$

(*iii*) $y = \dfrac{x^3}{4} + \dfrac{C}{x}$ (*iv*) $y = 2x^2 + Cx$

4. (*i*) $y = \dfrac{1}{5} (2 \sin x - \cos x) + Ce^{-2x}$ (*ii*) $y = \dfrac{1}{2} (\sin x - \cos x) + Ce^x$

(*iii*) $y = \dfrac{1}{6}xe^{4x} - \dfrac{1}{36}e^{4x} + Ce^{-2x}$ (*iv*) $y = \dfrac{1}{5} (2 \sin 2x + \cos 2x) + Ce^{-x}$

5. (*i*) $y = \log x + Ce^{-x}$ (*ii*) $y = \tan \dfrac{x}{2} + Ce^{-x}$

(*iii*) $y = \cos x + Ce^{-x}$ (*iv*) $yx^2 = (x^2 - 2) \sin x + 2x \cos x + C$

6. (*i*) $y = x^3 + Cx$ (*ii*) $y = \frac{6}{7}x^3 + \frac{C}{\sqrt{x}}$

(*iii*) $y = \frac{x^3}{4} + \frac{C}{x}$ (*iv*) $y = \frac{6}{7}x^3 + \frac{C}{\sqrt{x}}$

7. (*i*) $y + 1 + \sin x = C\, e^{\sin x}$ (*ii*) $y = \frac{x-1}{x}e^x + \frac{C}{x}$

(*iii*) $y = \frac{1}{4}x^2 \log |x| - \frac{1}{16}x^2 + \frac{C}{x^2}$ (*iv*) $y = \sin x + \frac{C}{x}$

8. (*i*) $y \cos x = \frac{\cos 2x}{2} + C$ (*ii*) $y = 1 - \frac{x - C}{\sec x + \tan x}$

(*iii*) $y \cos x = \frac{e^x}{2}(\sin x + \cos x) + C$ (*iv*) $y = \tan x - 1 + Ce^{-\tan x}$

9. (*i*) $y = x \log |x| - 1 + Cx$ (*ii*) $(1 + x^2)y = \log |\sin x| + C$

(*iii*) $y = \frac{x^2}{4} + \frac{C}{x^2}$ (*iv*) $y = \tan^{-1} x - 1 + Ce^{-\tan^{-1} x}$

10. (*i*) $y \log x = (\log x)^2 + C$ (*ii*) $ye^{2\sqrt{x}} = 2\sqrt{x} + C$

(*iii*) $y(1 + x^3) = \frac{1}{2}(x - \sin x \cos x) + C$ (*iv*) $ye^{\tan^{-1} x} = \frac{1}{2}e^{2\tan^{-1} x} + C$

11. (*i*) $y(x^2 + 1)^2 = \tan^{-1} x + C$ (*ii*) $y(x^2 + 1)^2 = -x + C$

(*iii*) $y = \frac{1}{2}\log x + \frac{C}{\log x}$ (*iv*) $y = x^2 + C\cos x$

12. (*i*) $y = \frac{x}{1-a} - \frac{1}{a} + Cx^a$ (*ii*) $y = -\frac{1}{2}\cos 2x \sec x + C \sec x$

(*iii*) $y = \cos x + C\cos^2 x$ (*iv*) $y = -\frac{1}{2}\cos 2x \operatorname{cosec} x + C \operatorname{cosec} x$

13. (*i*) $y \log x = -\frac{2\log x}{x} - \frac{2}{x} + C$ (*ii*) $y(x^2 - 1) = \log\left|\frac{x-1}{x+1}\right| + C$

14. (*i*) $y(x^2 + 1) = \frac{x\sqrt{x^2+4}}{2} + 2\log |x + \sqrt{x^2+4}| + C$

(*ii*) $y = \frac{1}{2}\log x + \frac{C}{\log x}$ (*iii*) $y \sec x = (x^2 - 2)\sin x + 2x\cos x + C$

(*iv*) $(1 + \sin x)\, y + \frac{x^2}{2} = C$

15. (*i*) $y = \frac{1}{3}\sin^2 x + C \operatorname{cosec} x$ (*ii*) $y = x^2 + C\operatorname{cosec} x$

(*iii*) $y = (1 + x^2)(x + \tan^{-1} x + C)$ (*iv*) $y = a + C\sqrt{1 - x^2}$

16. (*i*) $y = xe^{-1} - e^{-x}$ (*ii*) $y \sin x = 2x^2 - \pi^2/2$

(*iii*) $y = \sin x$ (*iv*) $y = (x^2 + 1)\sin x$

17. (*i*) $y = \cos x - 2\cos^2 x$ (*ii*) $y = \tan x - 1 + 2e^{-\tan x}$

(*iii*) $y = \frac{1}{2}(\sin x - \cos x) + \frac{3}{2}e^x$ (*iv*) $4y = x^2$

18. (*i*) $(1 + x^2)y = \tan^{-1} x - \pi/4$ (*ii*) $y = x^2 - \frac{\pi^2}{4\sin x}$

(*iii*) $xy + \tan^{-1} x = \frac{\pi}{4}$ (*iv*) $y = \cos x - 2\cos^2 x.$

NOTES

Hints

18. (*ii*) Here, I.F. = sin x.

$\therefore$ Solution is $y \sin x = \int (2x + x^2 \cot x) \sin x \, dx + C$.

$$\int (2x + x^2 \cot x) \sin x \, dx = \int 2x \sin x \, dx + \int x^2 \cos x \, dx$$

$$= \int (\sin x) 2x \, dx + \int x^2 \cos x \, dx$$

$$= \left((\sin x)x^2 - \int (\cos x)x^2 \, dx\right) + \int x^2 \cos x \, dx = x^2 \sin x.$$

(*iii*) We have $(1 + y + x^2y)dx + (x + x^3)dy = 0$.

$\Rightarrow$ $(1 + y(1 + x^2))dx + x(1 + x^2)dy = 0$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{1 + y(1 + x^2)}{x(1 + x^2)} = -\frac{1}{x(1 + x^2)} - \left(\frac{1}{x}\right)y.$$

Solution of $\frac{dy}{dx}$ + Py = Qy^n, where P and Q are Functions of x or Constants, by Reducing it to a Linear Differential Equation

Consider the differential equation $\frac{dy}{dx} + Py = Qy^n$, ...(1)

where P and Q are functions of x or constants and $n \neq 0, 1$.

Equation (1) is known as **'Bernoulli's equation'**.

Dividing (1) by y^n, we get $\quad y^{-n}\frac{dy}{dx} + Py^{-n+1} = Q$. ...(2)

Let $\quad z = y^{-n+1}$.

$$\therefore \quad \frac{dz}{dx} = (-n + 1)y^{-n}\frac{dy}{dx} \quad \text{or} \quad y^{-n}\frac{dy}{dx} = \frac{1}{1-n} \cdot \frac{dz}{dx}$$

$$\therefore \quad (2) \Rightarrow \frac{1}{1-n} \cdot \frac{dz}{dx} + Pz = Q \quad \Rightarrow \quad \frac{dz}{dx} + P(1-n)z = Q(1-n). \quad ...(3)$$

(3) is a linear differential equation with z as the dependent variable.

Working Steps for Solving $\frac{dy}{dx}$ + Py = Qy^n

Step I. Divide the equation by y^n and get $\quad y^{-n}\frac{dy}{dx} + Py^{-n+1} = Q$ ...(1)

Step II. Put $z = y^{-n+1}$. $\therefore \quad \frac{dz}{dx} = (-n + 1)y^{-n}\frac{dy}{dx}$ or $y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dz}{dx}$.

Putting the values of y^{-n+1} and $y^{-n}\frac{dy}{dx}$ in (1), we get

$$\frac{dz}{dx} + P(1-n)z = Q(1-n). \quad ...(2)$$

Step III. (2) is a linear differential equation with dependent variable z.

NOTES

Example 58. *Solve:* $\dfrac{dy}{dx} + \dfrac{2}{3}y = \dfrac{x}{\sqrt{y}}$.

Solution. We have $\dfrac{dy}{dx} + \dfrac{2}{3}y = \dfrac{x}{\sqrt{y}}$. ...(1)

This is a **Bernoulli's equation**.

Multiplying (1) by $\sqrt{y}$, we get $\sqrt{y}\dfrac{dy}{dx} + \dfrac{2}{3}y^{3/2} = x$. ...(2)

Let $z = y^{3/2}$. $\therefore \quad \dfrac{dz}{dx} = \dfrac{3}{2}y^{1/2}\dfrac{dy}{dx}$ or $\sqrt{y}\dfrac{dy}{dx} = \dfrac{2}{3}\dfrac{dz}{dx}$

$\therefore$ (2) $\Rightarrow$ $\dfrac{2}{3}\dfrac{dz}{dx} + \dfrac{2}{3}z = x \Rightarrow \dfrac{dz}{dx} + z = \dfrac{3}{2}x$ ...(3)

(3) is a linear differential equation with z as the dependent variable.

Here P = 1 and Q = $\dfrac{3}{2}x$.

$\therefore \quad \int P\,dx = \int 1 \cdot dx = x$ and we have I.F. $= e^{\int P\,dx} = e^x$

The solution of (3) is $z(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + C$.

$$\Rightarrow \quad ze^x = \int \frac{3}{2}x \cdot e^x\,dx + C \Rightarrow y^{3/2}e^x = \frac{3}{2}\left[xe^x - \int 1 \cdot e^x\,dx\right] + C$$

$$\Rightarrow \quad y^{3/2}e^x = \frac{3}{2}(x-1)e^x + C \quad \Rightarrow \quad \boldsymbol{y^{3/2} = \frac{3}{2}(x-1) + Ce^{-x}}.$$

Example 59. *Solve:* $y(x^2y + e^x)dx - e^x\,dy = 0$.

Solution. We have $y(x^2y + e^x)dx - e^x\,dy = 0$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y(x^2y + e^x)}{e^x} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{x^2y^2}{e^x} + y$$

$$\Rightarrow \quad \frac{dy}{dx} + (-1)y = \left(\frac{x^2}{e^x}\right)y^2 \qquad ...(1)$$

This is a **Bernoulli's equation**.

Dividing (1) by y^2, we get $\dfrac{1}{y^2}\dfrac{dy}{dx} + (-1)\dfrac{1}{y} = \dfrac{x^2}{e^x}$. ...(2)

Let $z = \dfrac{1}{y}$. $\therefore \quad \dfrac{dz}{dx} = (-1)y^{-2}\dfrac{dy}{dx}$ or $\dfrac{1}{y^2}\dfrac{dy}{dx} = -\dfrac{dz}{dx}$

$\therefore$ (2) $\Rightarrow$ $-\dfrac{dz}{dx} + (-1)z = \dfrac{x^2}{e^x} \Rightarrow \dfrac{dz}{dx} + 1 \cdot z = -\dfrac{x^2}{e^x}$ ...(3)

(3) is a linear differential equation with dependent variable z.

Here P = 1 and Q = $-\dfrac{x^2}{e^x}$.

$\therefore \quad \int P\,dx = \int 1\,dx = x$ and we have I.F. $= e^{\int P\,dx} = e^x$

NOTES

The solution of (3) is $z(\text{I.F.}) = \int \text{Q(I.F.)}\, dx + \text{C}.$

$$\Rightarrow \quad ze^x = \int \left(-\frac{x^2}{e^x}\right) e^x\, dx + \text{C}$$

$$\Rightarrow \quad \frac{1}{y} e^x = -\int x^2\, dx + \text{C} \quad \Rightarrow \quad \frac{1}{y} e^x = -\frac{x^3}{3} + \text{C}.$$

EXERCISE L

Solve the following differential equations:

1. $\frac{dy}{dx} + \frac{y}{x} = y^2$ **2.** $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$ **3.** $\frac{dy}{dx} + \frac{x}{1-x^2} y = x\sqrt{y}$

4. $\frac{dy}{dx} + \frac{2}{x} y = \frac{y^3}{x^3}$ **5.** $\frac{dy}{dx} + xy = y^2 e^{\frac{1}{2}x^2} \sin x$ **6.** $\frac{dy}{dx} + xy = xy^5$.

Answers

1. $\frac{1}{xy} + \log x = \text{C}$ **2.** $2x - y = \text{C}x^2 y$

3. $3\sqrt{y} - x^2 + 1 = \text{C}(1-x^2)^{1/4}$ **4.** $\frac{1}{y^2x^4} = \frac{1}{3x^6} + \text{C}$

5. $e^{-\frac{1}{2}x^2} = y(\cos x + \text{C})$ **6.** $\frac{1}{y^4} = 1 + \text{C}e^{2x^2}$.

Solution of $f'(y)\frac{dy}{dx} + \text{P}f(y) = \text{Q}$, where P and Q are Functions of x or Constants and $f(y)$ is Some Function of y, by Reducing it to a Linear Differential Equation

Consider the differential equation $f'(y)\frac{dy}{dx} + \text{P}f(y) = \text{Q}$, ...(1)

where P and Q are functions of x or constants and $f(y)$ is some function of y.

Let $z = f(y)$. $\therefore \quad \frac{dz}{dx} = f'(y)\frac{dy}{dx}$

$\therefore$ (1) $\Rightarrow \quad \frac{dz}{dx} + \text{P}z = \text{Q}.$ ...(2)

(2) is a *linear differential equation* with z as the dependent variable.

Working Steps for Solving $f'(y)\frac{dy}{dx} + \text{P}f(y) = \text{Q}$

Step I Put $z = f(y)$. $\therefore \quad \frac{dz}{dx} = f'(y)\frac{dy}{dx}$

Step II Put the values of $f(y)$ and $f'(y)\frac{dy}{dx}$ in the given differential equation and get $\frac{dz}{dx} + \text{P}z = \text{Q}$.

This is a linear differential equation with dependent variable z.

SOLVED EXAMPLES

NOTES

Example 60. *Solve:* $\dfrac{dy}{dx} + \dfrac{1}{x} = \dfrac{e^y}{x^2}$.

Solution. We have $\dfrac{dy}{dx} + \dfrac{1}{x} = \dfrac{e^y}{x^2}$. ...(1)

Dividing (1) by e^y, we get $e^{-y}\dfrac{dy}{dx} + \dfrac{1}{x}\cdot e^{-y} = \dfrac{1}{x^2}$.

$\Rightarrow$ $$-e^{-y}\frac{dy}{dx} + \left(-\frac{1}{x}\right)e^{-y} = -\frac{1}{x^2} \qquad ...(2)$$

(2) is a differential equation of the form $f'(y)\dfrac{dy}{dx} + \text{P}f(y) = \text{Q}$, where $f(y) = e^{-y}$.

Let $z = f(y) = e^{-y}$. $\therefore$ $\dfrac{dz}{dx} = -e^{-y}\dfrac{dy}{dx}$

$\therefore$ (2) $\Rightarrow$ $$\frac{dz}{dx} + \left(-\frac{1}{x}\right)z = -\frac{1}{x^2}. \qquad ...(3)$$

(3) is a linear differential equation with z as the dependent variable.

Here $\text{P} = -\dfrac{1}{x}$ and $\text{Q} = -\dfrac{1}{x^2}$.

$\therefore$ $$\int \text{P}\,dx = \int -\frac{1}{x}\,dx = -\log x = \log\frac{1}{x} \qquad \text{(Assuming } x > 0)$$

$\therefore$ $$\text{I.F.} = e^{\int \text{P}\,dx} = e^{\log\frac{1}{x}} = \frac{1}{x}$$

The solution of (3) is $z(\text{I.F.}) = \int \text{Q(I.F.)}\,dx + \text{C}$.

$\Rightarrow$ $$e^{-y}\cdot\frac{1}{x} = \int -\frac{1}{x^2}\cdot\frac{1}{x}\,dx + \text{C} \quad \Rightarrow \quad \frac{e^{-y}}{x} = -\frac{x^{-2}}{-2} + \text{C}$$

$\Rightarrow$ $$\frac{e^{-y}}{x} = \frac{1}{2x^2} + \text{C} \quad \Rightarrow \quad \mathbf{2xe^{-y} = 1 + 2Cx^2}.$$

Example 61. *Solve:* $\sin y\,\dfrac{dy}{dx} = \cos y(1 - x\cos y)$.

Solution. We have $\sin y\,\dfrac{dy}{dx} = \cos y(1 - x\cos y)$.

$\Rightarrow$ $$\sin y\frac{dy}{dx} - \cos y = -x\cos^2 y \qquad ...(1)$$

Dividing (1) by $\cos^2 y$, we get $\dfrac{\sin y}{\cos^2 y}\dfrac{dy}{dx} - \dfrac{\cos y}{\cos^2 y} = -x$.

$\Rightarrow$ $$\sec y\tan y\frac{dy}{dx} + (-1)\sec y = -x \qquad ...(2)$$

(2) is a differential equation of the form $f'(y)\dfrac{dy}{dx} + \text{P}f(y) = \text{Q}$, where $f(y) = \sec y$.

Let $z = f(y) = \sec y$. $\therefore$ $\dfrac{dz}{dx} = \sec y\tan y\dfrac{dy}{dx}$

$\therefore$ (2) $\Rightarrow$ $$\frac{dz}{dx} + (-1)z = -x \qquad ...(3)$$

(3) is a linear differential equation with z as the dependent variable.

Here $\text{P} = -1$ and $\text{Q} = -x$.

NOTES

$$\therefore \quad \int P\,dx = \int -1 \cdot dx = -x \quad \text{and we have} \quad \text{I.F.} = e^{\int P\,dx} = e^{-x}.$$

The solution of (3) is $\quad z(\text{I.F.}) = \int Q(\text{I.F.})\,dx + C.$

$$\Rightarrow \quad (\sec y)e^{-x} = \int -x \cdot e^{-x}\,dx + C$$

$$\Rightarrow \quad e^{-x} \sec y = -\left[x \cdot \frac{e^{-x}}{-1} - \int 1 \cdot \frac{e^{-x}}{-1}\,dx\right] + C$$

$$\Rightarrow \quad e^{-x} \sec y = xe^{-x} - \frac{e^{-x}}{-1} + C \quad \Rightarrow \quad \mathbf{\sec y = x + 1 + Ce^{x}}.$$

Remark. Please note carefully the placing of cos y on the LHS in equation (1). The placing of $x \cos^2 y$ on the LHS of (1) will not reduce the given differential equation to the desired form.

EXERCISE M

Solve the following differential equations:

1. $(1+x)\dfrac{dy}{dx} + 1 = e^{x-y}$

2. $\dfrac{dy}{dx} + \dfrac{1}{x}y = y^3$

3. $\dfrac{dy}{dx} = \dfrac{x^2 + y^2 + 1}{2xy}$

4. $2 \tan y \dfrac{dy}{dx} + x \sin^2 y = x^3 \cos^2 y$

5. $\dfrac{dy}{dx} + \dfrac{1}{x} \tan y = \dfrac{1}{x^2} \tan y \sin y$

6. $\dfrac{dy}{dx} - \dfrac{y}{x} \log y = \dfrac{y}{x^2 (\log y)^2}$.

Answers

1. $e^y(1 + x) = e^x + C$

2. $2xy^2 + Cx^2y^2 = 1$

3. $y^2 + 1 = x^2 + Cx$

4. $\tan^2 y = x^2 - 2 + Ce^{-\frac{1}{2}x^2}$

5. $2x \operatorname{cosec} y = 1 + 2Cx^2$

6. $(\log y)^3 = -\dfrac{3}{4x} + Cx^3$.

Hints

3. We have $\dfrac{dy}{dx} = \dfrac{x^2 + y^2 + 1}{2xy}$.

$$\Rightarrow \quad 2y\frac{dy}{dx} = \frac{x^2 + y^2 + 1}{x} \quad \Rightarrow \quad 2y\frac{dy}{dx} = x + \frac{y^2}{x} + \frac{1}{x} \quad \Rightarrow \quad 2y\frac{dy}{dx} + \left(-\frac{1}{x}\right)(y^2 + 1) = x.$$

Solution of Linear differential equation $\dfrac{dy}{dx} + Px = Q$, where P and Q are functions of y or constants

Let $$\frac{dx}{dy} + Px = Q \qquad \ldots(1)$$

be a linear differential equation, where P and Q are functions of y or constants.

Multiplying both sides of (1) by $e^{\int P\,dy}$, we get

$$e^{\int P\,dy}\frac{dx}{dy} + e^{\int P\,dy}\,Px = Qe^{\int P\,dy}.$$

$$\Rightarrow \quad e^{\int P\,dy}\frac{dx}{dy} + \frac{d}{dy}(e^{\int P\,dy})\cdot x = Qe^{\int P\,dy} \qquad \left(\because \ \frac{d}{dy}\int P dy = P\right)$$

$$\Rightarrow \quad \frac{d}{dy}(xe^{\int P\,dy}) = Qe^{\int P\,dy}$$

$$\Rightarrow \quad \int \frac{d}{dy}(xe^{\int P\,dy})\,dy = \int (Qe^{\int P\,dy})dy + C$$

$$\Rightarrow \quad \boldsymbol{xe^{\int P\,dy} = \int (Qe^{\int P\,dy})dy + C.}$$

This is the general solution of linear differential equation (1).

$e^{\int P\,dy}$ is called the integrating factor (I.F.) of (1).

Thus, the solution of (1) can also be written as $x(\text{I.F.}) = \int Q(\text{I.F.})\ dy + C$.

NOTES

Working Steps for Solving $\dfrac{dx}{dy} + Px = Q$

Step I. Identify P and Q and make sure that these are functions of y or constants.

Step II. Evaluate $\int P\,dy$.

Step III. Find $e^{\int P\,dy}$. This is the integrating factor (I.F.).

Step IV. Put the value of I.F. in the general solution $x(\text{I.F.}) = \int Q(\text{I.F.})\,dy + C$ and simplify it. This gives the general solution of the given differential equation.

SOLVED EXAMPLES

Example 62. *Solve:* $y\,dx + (x - y^3)\,dy = 0$.

Solution. We have $y\,dx + (x - y^3)dy = 0$.

$$\Rightarrow \quad y\frac{dx}{dy} + x - y^3 = 0 \quad \Rightarrow \quad \frac{dx}{dy} + \left(\frac{1}{y}\right)x = y^2 \qquad \text{...(1)}$$

(1) is a linear differential equation of the form $\dfrac{dx}{dy} + Px = Q$.

Here, $P = \dfrac{1}{y}$ and $Q = y^2$.

$$\int P\,dy = \int \frac{1}{y}\,dy = \log y \quad \therefore \quad \text{I.F.} = e^{\int P\,dy} = e^{\log y} = y$$

(Assuming $y > 0$)

The solution of (1) is $x(\text{I.F.}) = \int Q(\text{I.F.})\,dy + C$.

$$\Rightarrow \quad xy = \int y^2 y\,dy + C \quad \Rightarrow \quad \boldsymbol{xy = \frac{y^4}{4} + C.}$$

Example 63. *Solve:* $(x + 3y^2)\dfrac{dy}{dx} = y,\ y > 0$.

Solution. We have $(x + 3y^2)\dfrac{dy}{dx} = y$.

NOTES

$$\Rightarrow \qquad y\frac{dx}{dy} = x + 3y^2 \;\Rightarrow\; \frac{dx}{dy} = \frac{x}{y} + 3y \;\Rightarrow\; \frac{dx}{dy} + \left(-\frac{1}{y}\right)x = 3y \qquad ...(1)$$

This is a linear differential equation of the form $\frac{dx}{dy} + \text{P}x = \text{Q}$.

Here, $\text{P} = -\frac{1}{y}$ and $\text{Q} = 3y$.

$$\int \text{P}\,dy = \int -\frac{1}{y}\,dy = -\log y = \log y^{-1} = \log\frac{1}{y} \qquad (y > 0 \;\Rightarrow\; |y| = y)$$

$$\therefore \qquad \text{I.F.} = e^{\int \text{P}\,dy} = e^{\log 1/y} = \frac{1}{y}$$

The solution of (1) is $x(\text{I.F.}) = \int \text{Q(I.F.)}\,dy + \text{C}$.

$$\Rightarrow \qquad x\left(\frac{1}{y}\right) = \int 3y\left(\frac{1}{y}\right)dy + \text{C} \;\Rightarrow\; \boldsymbol{\frac{x}{y} = 3y + \text{C}}.$$

Example 64. *Solve:* $y^2\frac{dx}{dy} + x - \frac{1}{y} = 0$.

Solution. We have $y^2\frac{dx}{dy} + x - \frac{1}{y} = 0$.

$$\Rightarrow \qquad y^2\frac{dx}{dy} + x = \frac{1}{y} \;\Rightarrow\; \frac{dx}{dy} + \left(\frac{1}{y^2}\right)x = \frac{1}{y^3} \qquad ...(1)$$

(1) is a linear differential equation of the form $\frac{dx}{dy} + \text{P}x = \text{Q}$.

Here, $\text{P} = \frac{1}{y^2}$ and $\text{Q} = \frac{1}{y^3}$.

$$\int \text{P}\,dy = \int \frac{1}{y^2}\,dy = -\frac{1}{y} \qquad \therefore \quad \text{I.F.} = e^{\int \text{P}\,dy} = e^{-1/y}$$

The solution of (1) is $x(\text{I.F.}) = \int \text{Q(I.F.)}\,dy + \text{C}$.

$$\Rightarrow \qquad xe^{-1/y} = \int \frac{1}{y^3}e^{-1/y}\,dy + \text{C} \qquad ...(2)$$

Let $\text{I} = \int \frac{1}{y^3}e^{-1/y}\,dy$

Let $z = -\frac{1}{y} \quad \therefore \quad y = -\frac{1}{z}$ and $dy = \frac{1}{z^2}\,dz$

$$\therefore \qquad \text{I} = \int -z^3e^z \,.\, \frac{1}{z^2}\,dz = -\int ze^z\,dz$$

$$= -\left[ze^z - \int 1\,.\,e^z\,dz\right] = -ze^z + e^z = e^z(1 - z) = e^{-1/y}\left(1 + \frac{1}{y}\right)$$

$$\therefore \quad (2) \;\Rightarrow\; xe^{-1/y} = e^{-1/y}\left(1 + \frac{1}{y}\right) + \text{C} \;\Rightarrow\; \boldsymbol{x = 1 + \frac{1}{y} + \text{C}e^{1/y}}.$$

Example 65. *Solve:* $(1 + y^2) + (x - e^{\tan^{-1} y})\frac{dy}{dx} = 0$.

Solution. We have $(1 + y^2) + (x - e^{\tan^{-1} y})\frac{dy}{dx} = 0$.

$$\Rightarrow \quad (1 + y^2)\frac{dx}{dy} + x - e^{\tan^{-1} y} = 0 \quad \Rightarrow \quad \frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{e^{\tan^{-1} y}}{1+y^2} \qquad \ldots(1)$$

NOTES

Equation (1) is a linear differential equation of the form $\frac{dx}{dy} + \text{P}x = \text{Q}$.

Here $\quad \text{P} = \frac{1}{1+y^2} \quad$ and $\quad \text{Q} = \frac{e^{\tan^{-1} y}}{1+y^2}$.

$$\int \text{P}\, dy = \int \frac{1}{1+y^2} dy = \tan^{-1} y \quad \therefore \quad \text{I.F.} = e^{\int \text{P}\, dy} = e^{\tan^{-1} y}$$

The solution of (1) is $\quad x(\text{I.F.}) = \int \text{Q(I.F.)}\, dy + \text{C}$.

$$\Rightarrow \quad x e^{\tan^{-1} y} = \int \frac{e^{\tan^{-1} y}}{1+y^2} \cdot e^{\tan^{-1} y}\, dy + \text{C}$$

$$\Rightarrow \quad x e^{\tan^{-1} y} = \int e^{2z}\, dz + \text{C}, \text{ where } z = \tan^{-1} y$$

$$\Rightarrow \quad x e^{\tan^{-1} y} = \frac{e^{2z}}{2} + \text{C} = \frac{e^{2\tan^{-1} y}}{2} + \text{C} \quad \Rightarrow \quad \mathbf{2x e^{\tan^{-1} y} = e^{2\tan^{-1} y} + 2C}.$$

Example 66. *Solve: $y\, dx - (x + 2y^2)dy = 0$, $y > 0$ given that $y = 1$ when $x = 2$.*

Solution. We have $\quad y\, dx - (x + 2y^2)dy = 0$.

$$\Rightarrow \quad y\frac{dx}{dy} = x + 2y^2 \quad \Rightarrow \quad \frac{dx}{dy} = \frac{x}{y} + 2y \Rightarrow \frac{dx}{dy} + \left(-\frac{1}{y}\right)x = 2y \qquad \ldots(1)$$

(1) is a linear differential equation of the form $\frac{dx}{dy} + \text{P}x = \text{Q}$.

Here $\quad \text{P} = -\frac{1}{y} \quad$ and $\quad \text{Q} = 2y$.

$$\int \text{P}\, dy = \int -\frac{1}{y}\, dy = -\log y = \log y^{-1} = \log \frac{1}{y} \qquad (y > 0 \Rightarrow |y| = y)$$

$$\therefore \quad \text{I.F.} = e^{\int \text{P}\, dy} = e^{\log \frac{1}{y}} = \frac{1}{y}$$

The solution of (1) is $x(\text{I.F.}) = \int \text{Q(I.F.)}\, dy + \text{C}$.

$$\Rightarrow \quad x\left(\frac{1}{y}\right) = \int 2y\left(\frac{1}{y}\right) dy + \text{C} \quad \Rightarrow \quad \frac{x}{y} = 2y + \text{C} \qquad \ldots(2)$$

Now $y = 1$ when $x = 2$. $\quad \therefore \quad (2) \quad \Rightarrow \quad \frac{2}{1} = 2(1) + \text{C} \quad$ or $\quad \text{C} = 0$

$\therefore$ Using (2), the required solution is $\frac{x}{y} = 2y + 0$ *i.e.,* $\mathbf{x = 2y^2}$.

EXERCISE N

Solve the following differential equations:

1. (*i*) $y\,dx - (x + 2y^2)dy = 0$ (*ii*) $(x + 2y^3)\dfrac{dy}{dx} = y,\ y > 0$

(*iii*) $(3y^2 - x)dy = y\,dx,\ y > 0$ (*iv*) $y^2 + \left(x - \dfrac{1}{y}\right)\dfrac{dy}{dx} = 0$

2. (*i*) $(1 + y^2)dx = (\tan^{-1} y - x)dy$ (*ii*) $(1 + y^2) + (2xy - \cot y)\dfrac{dy}{dx} = 0$

(*iii*) $(2x - 10y^3)\dfrac{dy}{dx} + y = 0$ (*iv*) $(x + y)\dfrac{dy}{dx} = 1.$

3. (*i*) $(1 + y^2)dx + (x - e^{-\tan^{-1} y})dy = 0,\ y(0) = 0$

(*ii*) $(1 + y^2)dx = (\tan^{-1} y - x)dy,\ y(1) = 0$

(*iii*) $(x - \sin y)dy + (\tan y)dx = 0,\ y(0) = 0.$

(*iv*) $\dfrac{dx}{dy} + x\cot y = 2y + y^2 \cot y,\ \ (y \neq 0),\ y(0) = \dfrac{\pi}{2}.$

Answers

1. (*i*) $x = 2y^2 + Cy$ (*ii*) $x = y^3 + Cy$

(*iii*) $xy = y^3 + C$ (*iv*) $x = 1 + y^{-1} + Ce^{1/y}$

2. (*i*) $x = \tan^{-1} y - 1 + Ce^{-\tan^{-1} y}$ (*ii*) $x(1 + y^2) = \log|\sin y| + C$

(*iii*) $x = 2y^3 + Cy^{-2}$ (*iv*) $x + y - 1 = Ce^y$

3. (*i*) $xe^{\tan^{-1} y} = \tan^{-1} y$ (*ii*) $(x - \tan^{-1} y + 1)e^{\tan^{-1} y} = 2$

(*iii*) $2x = \sin y$ (*iv*) $4(x - y^2)\sin y + \pi^2 = 0.$

SUMMARY

1. An equation involving independent and dependent variables and at least one derivative/differential of these variables called a **differential equation**.
2. The **order** of a differential equation is the order of the derivative of the highest order, occurring in the differential equation.
3. The **degree** of a differential equation is defined if it can be written as a polynomial equation in the derivatives and for such a differential equation its degree is given by the highest power of the highest order derivative appearing in it, provided the derivatives are made free from radicals and fractions.
4. A differential equation is said to be **linear**, if the dependent variable and its derivatives occur only in the first degree and are not multiplied together.
5. (*i*) A **solution** of a differential equation is a functional relation between the variables involved which satisfies the given differential equation.

   (*ii*) A solution of a differential equation is called the **general solution (or complete solution)**, if it contains as many arbitrary constants as the order of the differential equation.

   (*iii*) A solution obtained by giving particular values to arbitrary constants in the general solution of a differential equation is called a **particular solution** of the differential equation, under consideration.

6. (*i*) If $\frac{dy}{dx} = f(x)$, then $dy = f(x)\,dx$. $\therefore \int 1 \,.\, dy = \int f(x)\,dx + C$.

This represents the general solution of the given differential equation.

(*ii*) If $\frac{dy}{dx} = g(y)$, then $\frac{dy}{g(y)} = f(x)$. $\therefore \int \frac{dy}{g(y)} = \int 1 \,.\, f(x) + C$.

This represents the general solution of the given differential equation.

(*iii*) If $\frac{dy}{dx} = f(x)\,g(y)$, then $\frac{dy}{g(y)} = f(x)\,dx$. $\therefore \int \frac{1}{g(y)}\,dy = \int f(x)\,dx + C$.

This represents the general solution of the given differential equation.

7. If $\frac{dy}{dx} = f(ax + by + c)$, then $z = ax + by + c$ reduces the given differentiable equation to 'variable separable' form.

8. If $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ is a homogeneous equation, then $y = vx$ reduces the given differential equation to 'variable separable' form.

9. If $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$ and $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ then put $x = X + h$ and $y = Y + k$ where h and k are constant such that $a_1h + b_1k + c_1 = 0$, $a_2h + b_2k + c_2 = 0$. The substitution $Y = VX$ reduces the resultant equation to 'variable separable' form.

10. If $\frac{dy}{dx} + Py = Q$ is a linear differential equation, where P and Q are functions of x or constants, then $ye^{\int P\,dx} = \int (Qe^{\int P\,dx})\,dx + C$

is the general solution of the given differential equation.

11. A differential equation of the form $\frac{dy}{dx} + Py = Qy^n$, where $n \neq 0, 1$ and P and Q are functions of x or constants, is solved by putting $z = y^{-n+1}$. This substitution reduces the given differential equation to a linear differential equation.

12. A differential equation of the form $f'(y)\frac{dy}{dx} + f(y)P = Q$, where P and Q are functions of x or constants, is solved by putting $z = f(y)$. This substitution reduces the given differential equation to a linear differential equation.

13. If $\frac{dx}{dy} + Px = Q$ is a linear differential equation, where P and Q are functions of y or constants, then $x\,e^{\int P\,dy} = \int (Q\,e^{\int P\,dy})\,dy + C$ is the general solution of the given differential equation.

NOTES

UNIT II

2. EXACT DIFFERENTIAL EQUATIONS

STRUCTURE

INTRODUCTION

A differential equation obtained from its primitive directly by differentiation, without any operation of multiplication, elimination or reduction etc. is said to be an exact differential equation.

Thus a differential equation of the form M (x, y) dx + N (x, y) $dy = 0$ is an exact differential equation if it can be obtained directly by differentiating the equation u $(x, y) = c$, which is its primitive.

i.e., if $$du = \text{M}dx + \text{N}dy.$$

For example, the equation $xdx + ydy = 0$ is an exact differential equation, as it can be obtained from its primitive $x^2 + y^2 = c^2$ directly by differentiation.

THEOREM

The necessary and sufficient condition for the differential equation Mdx + Ndy = 0 to be exact is

$$\frac{\partial \text{M}}{\partial y} = \frac{\partial \text{N}}{\partial x}.$$

The condition is necessary

The equation Mdx + Ndy = 0 will be exact, if $du = \text{M}dx + \text{N}dy$

But $$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$\therefore \qquad \text{M}dx + \text{N}dy = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

Equating co-efficients of dx and dy, we get

$$\text{M} = \frac{\partial u}{\partial x} \quad \text{and} \quad \text{N} = \frac{\partial u}{\partial y}$$

$\therefore$ $$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

But $$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

$\therefore$ $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

which is the necessary condition of exactness.

The condition is sufficient.

Let $$u = \int_{y \text{ constant}} M dx$$

$\therefore$ $$\frac{\partial u}{\partial x} = M \quad \text{and} \quad \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial M}{\partial y}$$

But $$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} \quad \text{and} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ $$\frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

Integrating both sides w.r.t. x treating y as constant, we have $N = \dfrac{\partial u}{\partial y} + f(y)$

$\therefore$ $$M dx + N dy = \frac{\partial u}{\partial x} dx + \left\{\frac{\partial u}{\partial y} + f(y)\right\} dy \qquad \left[\because \; M = \frac{\partial u}{\partial x}, N = \frac{\partial u}{\partial y} + f(y)\right]$$

$$= \left(\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy\right) + f(y)\, dy = du + f(y)\, dy = d[u + \textstyle\int f(y)\, dy]$$

which shows that $M dx + N dy$ is an exact differential and hence $M dx + N dy = 0$ is an exact differential equation.

Note. Since $M dx + N dy = d[u + \int f(y)\, dy]$

$\therefore$ $M dx + N dy = 0 \;\Rightarrow\; d[u + \int f(y)\, dy] = 0$

Integrating, $u + \int f(y)\, dy = c$

But $u = \displaystyle\int_{y \text{ constant}} M dx$ and $f(y)$ = terms of N not containing x

Hence the solution of $M dx + N dy = 0$ is

$$\int_{y \text{ constant}} \mathbf{M dx} + \int \textbf{(terms of N not containing } x\textbf{)}\, dy = c.$$

SOLVED EXAMPLES

Example 1. *Solve* $(5x^4 + 3x^2y^2 - 2xy^3)\, dx + (2x^3y - 3x^2y^2 - 5y^4)\, dy = 0.$

Sol. Here $M = 5x^4 + 3x^2y^2 - 2xy^3$ and $N = 2x^3y - 3x^2y^2 - 5y^4$

$\therefore$ $$\frac{\partial M}{\partial y} = 6x^2y - 6xy^2 = \frac{\partial N}{\partial x}$$

Thus the given equation is exact and its solution is

$$\int_{y \text{ constant}} \mathrm{M}dx + \int (\text{terms of N not containing } x)\, dy = c$$

i.e.,
$$\int_{y \text{ constant}} (5x^4 + 3x^2y^2 - 2xy^3)\, dx + \int -5y^4 dy = c$$

or
$$x^5 + x^3y^2 - x^2y^3 - y^5 = c.$$

Example 2. *Solve* $[\cos x \tan y + \cos (x + y)]\, dx + [\sin x \sec^2 y + \cos (x + y)]\, dy = 0.$

Sol. Here, $\mathrm{M} = \cos x \tan y + \cos (x + y)$

and $\mathrm{N} = \sin x \sec^2 y + \cos (x + y)$

$$\therefore \quad \frac{\partial \mathrm{M}}{\partial y} = \cos x \sec^2 y - \sin (x + y) = \frac{\partial \mathrm{N}}{\partial x}$$

Thus the given equation is exact and its solution is

$$\int_{y \text{ constant}} \mathrm{M}dx + \int (\text{terms of N not containing } x)\, dy = c$$

i.e.,
$$\int_{y \text{ constant}} [\cos x \tan y + \cos (x + y)]\, dx = 0$$

or
$$\sin x \tan y + \sin (x + y) = c.$$

Example 3. *Solve* $\dfrac{dy}{dx} + \dfrac{y \cos x + \sin y + y}{\sin x + x \cos y + x} = 0.$

Sol. The given equation can be written as

$$(y \cos x + \sin y + y)\, dx + (\sin x + x \cos y + x)\, dy = 0$$

Here $\mathrm{M} = y \cos x + \sin y + y$ and $\mathrm{N} = \sin x + x \cos y + x$

$$\therefore \quad \frac{\partial \mathrm{M}}{\partial y} = \cos x + \cos y + 1 = \frac{\partial \mathrm{N}}{\partial x}$$

Thus the given equation is exact and its solution is

$$\int_{y \text{ constant}} \mathrm{M}dx + \int (\text{terms of N not containing } x)\, dy = c$$

i.e.,
$$\int_{y \text{ constant}} (y \cos x + \sin y + y)\, dx = c$$

or
$$y \sin x + (\sin y + y)\, x = c.$$

EXERCISE A

Solve the following differential equations (1 to 22):

1. $(1 + 4xy + 2y^2)dx + (1 + 4xy + 2x^2)\, dy = 0$

2. $(3x^2 + 6xy^2)dx + (6x^2y + 4y^3)\, dy = 0$

3. $y(y^2 - 3x^2)dy + x(x^2 - 3y^2)\, dx = 0,\ y(0) = 1$

4. $(2x^3 - xy^2 - 2y + 3)\, dx - (x^2y + 2x)dy = 0$

5. $\dfrac{dy}{dx} + \dfrac{ax + hy + g}{hx + by + f} = 0$

6. $\left[\dfrac{y^2}{(y - x)^2} - \dfrac{1}{x}\right]dx + \left[\dfrac{1}{y} - \dfrac{x^2}{(x - y)^2}\right]dy = 0$

7. $xdy + ydx + \frac{xdy - ydx}{x^2 + y^2} = 0$ **8.** $xdx + ydy = \frac{a^2(xdy - ydx)}{x^2 + y^2}$

9. $dx = \frac{y}{1 - x^2y^2}dx + \frac{x}{1 - x^2y^2}dy$ **10.** $2x\left(1 + \sqrt{x^2 - y}\right)dx = \sqrt{x^2 - y}\ dy$

11. $(y \cos x + 1)\ dx + \sin x\ dy = 0$

12. (*i*) $\left[y\left(1 + \frac{1}{x}\right) + \cos y\right]dx + (x + \log x - x \sin y)\,dy = 0$

(*ii*) $\left[y\left(1 + \frac{1}{x}\right)\cos y\right]dx + (x + \log x)(\cos y - y \sin y)\,dy = 0$

13. $(2xy + y - \tan y)dx + (x^2 - x\tan^2 y + \sec^2 y)dy = 0$

14. $(1 + e^{x/y})dx + \left(1 - \frac{x}{y}\right)e^{x/y}dy = 0$ **15.** $e^y dx + (xe^y + 2y)\ dy = 0$

16. $ye^{xy}\ dx + (xe^{xy} + 2y)\ dy = 0$ **17.** $(y^2e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy = 0$

18. $(\sec x \tan x \tan y - e^x)dx + \sec x \sec^2 y\ dy = 0$

19. $(\sin x \cos y + e^{2x})\ dx + (\cos x \sin y + \tan y)\ dy = 0$

20. $(2xy\cos x^2 - 2xy + 1)dx + (\sin x^2 - x^2)dy = 0$

21. $e^x(\cos y\ dx - \sin y\ dy) = 0, y(0) = 0$

22. $\left[\cos x \log(2y - 8) + \frac{1}{x}\right]dx + \frac{\sin x}{y - 4}dy = 0, \quad y(1) = \frac{9}{2}$

23. Find the value of λ for which the differential equation $(xy^2 + \lambda x^2y)dx + (x + y)x^2dy = 0$, is exact. Hence solve it.

Answers

1. $(x + y)(1 + 2xy) = c$ **2.** $x^3 + 3x^2y^2 + y^4 = c$ **3.** $x^4 - 6x^2y^2 + y^4 = 1$

4. $x^4 - x^2y^2 - 4xy + 6x = c$ **5.** $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

6. $\frac{y^2}{y - x} + \log\frac{y}{x} = c$ **7.** $xy - \tan^{-1}\left(\frac{x}{y}\right) = c$

8. $x^2 + y^2 + 2a^2\tan^{-1}\left(\frac{x}{y}\right) = c$ **9.** $\log\frac{1 + xy}{1 - xy} - 2x = c$ **10.** $3x^2 + 2(x^2 - y)^{3/2} = c$

11. $y \sin x + x = c$ **12.** (*i*) $y(x + \log x) + x \cos y = c$ (*ii*) $y \cos y\ (x + \log x) = c$

13. $x^2y + xy - x \tan y + \tan y = c$ **14.** $x + ye^{x/y} = c$ **15.** $xe^y + y^2 = c$

16. $e^{xy} + y^2 = c$ **17.** $e^{xy^2} + x^4 - y^3 = c$ **18.** $\sec x \tan y - e^x = c$

19. $-\cos x \cos y + \frac{1}{2}e^{2x} + \log\sec y = c$ **20.** $y \sin x^2 - x^2y + x = c$

21. $e^x \cos y = 1$ **22.** $\sin x \log(2y - 8) + \log x = 0$

23. $\lambda = 3;\ \frac{1}{2}x^2y^2 + x^3y = c$.

EQUATIONS REDUCIBLE TO EXACT EQUATIONS

NOTES

Differential equations which are not exact can sometimes be made exact after multiplying by a suitable factor (a function of x and/or y) called the integrating factor.

For example, consider the equation $y\,dx - x\,dy = 0$ ...(1)

Here, $\quad M = y \quad$ and $\quad N = -x$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}, \text{ therefore the equation is not exact.}$$

(*i*) Multiplying the equation by $\frac{1}{y^2}$, it becomes

$$\frac{ydx - xdy}{y^2} = 0 \quad \text{or} \quad d\left(\frac{x}{y}\right) = 0 \qquad \text{which is exact.}$$

(*ii*) Multiplying the equation by $\frac{1}{x^2}$, it becomes

$$\frac{ydx - xdy}{x^2} = 0 \quad \text{or} \quad d\left(\frac{y}{x}\right) = 0 \qquad \text{which is exact.}$$

(*iii*) Multiplying the equation by $\frac{1}{xy}$, it becomes

$$\frac{dx}{x} - \frac{dy}{y} = 0 \quad \text{or} \quad d(\log x - \log y) = 0 \qquad \text{which is exact.}$$

$\therefore \quad \frac{1}{y^2}, \frac{1}{x^2}$ and $\frac{1}{xy}$ are integrating factors of (1).

If a differential equation has one integrating factor, it has an infinite number of integrating factors.

I.F. Found by Inspection

In a number of problems, a little analysis helps to find the integrating factor. The following differentials are useful in selecting a suitable integrating factor.

(*i*) $ydx + xdy = d(xy)$

(*ii*) $\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right)$

(*iii*) $\frac{ydx - xdy}{y^2} = d\left(\frac{x}{y}\right)$

(*iv*) $\frac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$

(*v*) $\frac{xdy - ydx}{xy} = d\left[\log\left(\frac{y}{x}\right)\right]$

(*vi*) $\frac{ydx + xdy}{xy} = d\,[\log(xy)]$

(*vii*) $\frac{xdx + ydy}{x^2 + y^2} = d\left[\frac{1}{2}\log(x^2 + y^2)\right]$

(*viii*) $\frac{xdy - ydx}{x^2 - y^2} = d\left(\frac{1}{2}\log\frac{x+y}{x-y}\right)$.

SOLVED EXAMPLES

Example 4. *Solve* $ydx - xdy + 3x^2y^2e^{x^3}\,dx = 0$.

Sol. Since $3x^2e^{x^3} = d(e^{x^3})$, the term $3x^2y^2e^{x^3}\,dx$ should not involve y^2.

This suggests that $\frac{1}{y^2}$ may be an I.F.

Multiplying throughout by $\frac{1}{y^2}$, we have $\frac{ydx - xdy}{y^2} + 3x^2 e^{x^3}\ dx = 0$

or $$d\left(\frac{x}{y}\right) + d\,(e^{x^3}) = 0, \text{ which is exact.}$$

Integrating, we get $\frac{x}{y} + e^{x^3} = c$, which is the required solution.

Example 5. *Solve* $xdy - ydx = x\sqrt{x^2 - y^2}\ dx$.

Sol. The given equation is $xdy - ydx = x^2 \sqrt{1 - \left(\frac{y}{x}\right)^2}\,dx$ or $\dfrac{\frac{xdy - ydx}{x^2}}{\sqrt{1-\left(\frac{y}{x}\right)^2}} = dx$

or $$d\left(\sin^{-1}\frac{y}{x}\right) = dx, \text{ which is exact.}$$

Integrating, we get $\sin^{-1}\frac{y}{x} = x + c$ or $y = x \sin(x + c)$, which is the required solution.

Example 6. *Solve:* $xdx + ydy = \dfrac{a^2(xdy - ydx)}{x^2 + y^2}$.

Sol. The given equation is $xdx + ydy - a^2 d\left(\tan^{-1}\frac{y}{x}\right) = 0$

Integrating, we get $\frac{x^2}{2} + \frac{y^2}{2} - a^2 \tan^{-1}\frac{y}{x} = c$

or $$x^2 + y^2 - 2a^2 \tan^{-1}\frac{y}{x} = C, \qquad \text{where } C = 2c.$$

EXERCISE B

Solve the following differential equations:

1. $xdy - ydx = (x^2 + y^2)\ dx$
2. $xdy - ydx = (x^2 + y^2)\ (dx + dy)$
3. $y(2xy + e^x)\ dx = e^x\ dy$
4. $(y \log y - 2xy)\ dx + (x + y)\ dy = 0$
5. $xdy - ydx = xy^2\ dx$
6. $xdy = (x^2y^2 - y)\ dx$
7. $(x + y)^2\left(x\frac{dy}{dx} + y\right) = xy\left(1 + \frac{dy}{dx}\right)$
8. $xdy - ydx = (4x^2 + y^2)dy$
9. $(y + y^2 \cos x)\ dx - (x - y^3)\ dy = 0.$

Answers

1. $\tan^{-1}\frac{y}{x} = x + c$
2. $\tan^{-1}\frac{y}{x} = x + y + c$
3. $x^2 + \frac{e^x}{y} = c$
4. $x \log y - x^2 + y = c$
5. $\frac{x}{y} + \frac{x^2}{2} = c$
6. $-\frac{1}{xy} = x + c$
7. $\log(xy) = -\frac{1}{x + y} + c$
8. $\frac{1}{2}\tan^{-1}\left(\frac{y}{2x}\right) = y + c$
9. $\frac{x}{y} + \sin x + \frac{y^2}{2} = c.$

NOTES

Hints

1. $\dfrac{xdy - ydx}{x^2 + y^2} = dx \Rightarrow d\left(\tan^{-1}\dfrac{y}{x}\right) = dx$

3. I.F. $= \dfrac{1}{y^2}$ and $\dfrac{ye^x dx - e^x dy}{y^2} = d\left(\dfrac{e^x}{y}\right)$

4. I.F. $= \dfrac{1}{y}$ and $\dfrac{x}{y}dy + \log y dx = d\,(x \log y)$

6. $\dfrac{xdy + ydx}{x^2y^2} = dx \Rightarrow \dfrac{d(xy)}{(xy)^2} = dx$

7. $\dfrac{xdy + ydx}{xy} = \dfrac{dx + dy}{(x + y)^2} \Rightarrow \dfrac{d(xy)}{xy} = \dfrac{d(x + y)}{(x + y)^2}$

8. $\dfrac{xdy - ydx}{4x^2 + y^2} = dy \Rightarrow \dfrac{(xdy - ydx)/x^2}{4 + (y/x)^2} = dy \Rightarrow \dfrac{d\left(\dfrac{y}{x}\right)}{4 + \left(\dfrac{y}{x}\right)^2} = dy$.

I.F. for a Homogeneous Equation

If Mdx + Ndy = 0 is a homogeneous equation in x and y, then $\dfrac{1}{\text{M}x + \text{N}y}$ is an I.F. provided Mx + Ny ≠ 0.

Note. If Mx + Ny consists of only one term, use the above method of I.F. otherwise, proceed by putting $y = vx$.

Example 7. *Solve:* $(x^2y - 2xy^2)\, dx - (x^3 - 3x^2y)\, dy = 0$.

Sol. The given equation is homogeneous in x and y with

$$\text{M} = x^2y - 2xy^2 \quad \text{and} \quad \text{N} = -x^3 + 3x^2y$$

Now, $\quad \text{M}x + \text{N}y = x^3y - 2x^2y^2 - x^3y + 3x^2y^2 = x^2y^2 \neq 0$

$\therefore \quad \text{I.F.} = \dfrac{1}{\text{M}x + \text{N}y} = \dfrac{1}{x^2y^2}$.

Multiplying throughout by $\dfrac{1}{x^2y^2}$, the given equation becomes

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0, \text{ which is exact.}$$

The solution is $\displaystyle\int_{y\text{ constant}} \left(\frac{1}{y} - \frac{2}{x}\right)dx + \int \frac{3}{y}\,dy = c$

or $\quad \dfrac{x}{y} - 2\log x + 3\log y = c$.

EXERCISE C

Solve the following differential equations:

1. $(xy - 2y^2)\, dx - (x^2 - 3xy)\, dy = 0$
2. $x^2y\, dx - (x^3 + y^3)\, dy = 0$
3. $(3xy^2 - y^3)\, dx - (2x^2y - xy^2)\, dy = 0$
4. $(x^2 - 3xy + 2y^2)\, dx + x(3x - 2y)dy = 0$.

Answers

1. $\dfrac{x}{y} - 2\log x + 3\log y = c$
2. $\log y - \dfrac{x^3}{3y^3} = c$
3. $3\log x - 2\log y + \dfrac{y}{x} = c$
4. $x^2 \log x + 3xy - y^2 = cx^2$

I.F. for an Equation of the Form $f_1(xy)\, ydx + f_2(xy)\, xdy = 0$.

If Mdx + Ndy = 0 is of the form $f_1(xy)\, ydx + f_2(xy)\, xdy = 0$, then $\frac{1}{Mx - Ny}$ is an I.F. provided $Mx - Ny \neq 0$.

Example 8. *Solve:* $y\,(xy + 2x^2y^2)\, dx + x(xy - x^2y^2)\, dy = 0$.

Sol. The given equation is of the form $f_1(xy)\, y\, dx + f_2(xy)\, x\, dy = 0$.

Here, $M = xy^2 + 2x^2y^3$ and $N = x^2y - x^3y^2$

Now, $Mx - Ny = x^2y^2 + 2x^3y^3 - x^2y^2 + x^3y^3 = 3x^3y^3 \neq 0$

$$\therefore \qquad \text{I.F.} = \frac{1}{Mx - Ny} = \frac{1}{3x^3y^3}$$

Multiplying throughout by $\frac{1}{3x^3y^3}$, the given equation becomes

$$\left(\frac{1}{3x^2y} + \frac{2}{3x}\right) dx + \left(\frac{1}{3xy^2} - \frac{1}{3y}\right) dy = 0$$

which is exact. The solution is $\displaystyle\int_{y\text{ constant}} \left(\frac{1}{3x^2y} + \frac{2}{3x}\right) dx + \int -\frac{1}{3y}\, dy = c$

or $$-\frac{1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = c$$

or $$-\frac{1}{xy} + 2\log x - \log y = C, \qquad \text{where } C = 3c.$$

EXERCISE D

Solve the following differential equations:

1. $(1 + xy)\, ydx + (1 - xy)xdy = 0$.
2. $(x^2y^2 + xy + 1)\, ydx + (x^2y^2 - xy + 1)xdy = 0$.
3. $y(2xy + 1)dx + x(1 + 2xy - x^3y^3)dy = 0$.
4. $(xy^2 + 2x^2y^3)dx + (x^2y - x^3y^2)\, dy = 0$.
5. $(y - xy^2)dx - (x + x^2y)\, dy = 0$.
6. $(xy \sin xy + \cos xy)\, ydx + (xy \sin xy - \cos xy)\, xdy = 0$.

Answers

1. $-\frac{1}{xy} + \log\left(\frac{x}{y}\right) = c$
2. $xy + \log\left(\frac{x}{y}\right) - \frac{1}{xy} = c$
3. $\frac{1}{x^2y^2} + \frac{1}{3x^3y^3} + \log y = c$
4. $-\frac{1}{xy} + 2\log x - \log y = c$
5. $\log\left(\frac{x}{y}\right) - xy = c$
6. $y \cos xy = cx$.

For the equation Mdx + Ndy = 0

(*i*) If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$, a function of x only, then $e^{\int f(x)dx}$ is an I.F.

(*ii*) If $\dfrac{\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}}{M}=g(y)$, a function of y only, then $e^{\int g(y)dy}$ is an I.F.

NOTES

SOLVED EXAMPLES

Example 9. *Solve:* $(xy^2 - e^{\frac{1}{x^3}})\,dx - x^2y\,dy = 0.$

Sol. Here, $M = xy^2 - e^{\frac{1}{x^3}}$ and $N = -x^2y$

$$\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=\frac{2xy-(-2xy)}{-x^2y}=-\frac{4}{x},\quad \text{which is a function of } x \text{ only.}$$

$$\therefore \qquad \text{I.F.} = e^{\int -\frac{4}{x}dx} = e^{-4\log x} = \frac{1}{x^4}$$

Multiplying throughout by $\dfrac{1}{x^4}$, we have $\left(\dfrac{y^2}{x^3}-\dfrac{1}{x^4}e^{\frac{1}{x^3}}\right)dx-\dfrac{y}{x^2}\,dy=0$

which is exact. The solution is $\displaystyle\int_{y\text{ constant}}\left(\frac{y^2}{x^3}-\frac{1}{x^4}e^{\frac{1}{x^3}}\right)dx=c$

or $$-\frac{y^2}{2x^2}+\frac{1}{3}\int -\frac{3}{x^4}e^{\frac{1}{x^3}}\,dx=c \quad\text{or}\quad -\frac{y^2}{2x^2}+\frac{1}{3}\int e^t\,dt=c,\quad \text{where } t=\frac{1}{x^3}$$

or $$-\frac{y^2}{2x^2}+\frac{1}{3}e^t=c \quad\text{or}\quad -\frac{3y^2}{x^2}+2e^{\frac{1}{x^3}}=C,\ \text{where } C=6c.$$

Example 10. *Solve:* $(xy^3 + y)\,dx + 2(x^2y^2 + x + y^4)\,dy = 0.$

Sol. Here, $M = xy^3 + y$ and $N = 2x^2y^2 + 2x + 2y^4$

$$\frac{\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}}{M}=\frac{4xy^2+2-3xy^2-1}{xy^3+y}=\frac{xy^2+1}{y(xy^2+1)}=\frac{1}{y}$$

which is a function of y only.

$$\therefore \qquad \text{I.F.} = e^{\int \frac{1}{y}dy} = e^{\log y} = y$$

Multiplying throughout by y, we have $(xy^4 + y^2)\,dx + 2\,(x^2y^3 + xy + y^5)\,dy = 0$

which is exact. The solution is $\displaystyle\int_{y\text{ constant}}(xy^4+y^2)\,dx+\int 2y^5\,dy=c$

or $$\frac{x^2y^4}{2}+xy^2+\frac{y^6}{3}=c.$$

EXERCISE E

NOTES

Solve the following differential equations:

1. $(x^2 + y^2 + x)\, dx + xydy = 0$
2. $(x^2 + y^2 + 1)\, dx - 2xydy = 0.$
3. $(x^2 + y^2 + 2x)dx + 2ydy = 0.$
4. $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)\, dy = 0.$
5. $\left(y + \frac{y^3}{3} + \frac{x^2}{2}\right)dx + \frac{1}{4}(x + xy^2)dy = 0.$
6. $(x \sec^2 y - x^2 \cos y)dy = (\tan y - 3x^4)dx.$
7. $(xye^{x/y} + y^2)\, dx - x^2e^{x/y}\, dy = 0.$
8. $(3xy - 2ay^2)dx + (x^2 - 2axy)dy = 0.$
9. $(x^4e^x - 2mxy^2)dx + 2mx^2ydy = 0.$
10. $y(2x^2y + e^x)dx = (e^x + y^3)dy.$
11. $ydx - xdy + \log x\, dx = 0.$
12. $(2x \log x - xy)\, dy + 2ydx = 0.$
13. $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)\, dy = 0.$
14. $y \log y\, dx + (x - \log y)\, dy = 0.$
15. $(xy^3 + y)dx + 2\,(x^2y^2 + x + y^4)dy = 0.$

Answers

1. $3x^4 + 6x^2y^2 + 4x^3 = c$
2. $x - \frac{y^2}{x} - \frac{1}{x} = c$
3. $e^x(x^2 + y^2) = c$
4. $\left(y + \frac{2}{y^2}\right)x + y^2 = c$
5. $3x^2y + x^4y^3 + x^6 = c$
6. $\frac{\tan y}{x} + x^3 - \sin y = c$
7. $e^{x/y} + \log x = c$
8. $x^2y(x - ay) = c$
9. $e^x + m\left(\frac{y}{x}\right)^2 = c$
10. $\frac{2x^3}{3} + \frac{e^x}{y} - \frac{y^2}{2} = c$
11. $1 + y + \log x = cx$
12. $2y \log x - \frac{1}{2}y^2 = c$
13. $x^3y^2 + \frac{x^2}{y} = c$
14. $x \log y - \frac{1}{2}(\log y)^2 = c$
15. $\frac{x^2y^4}{2} + xy^2 + \frac{y^6}{3} = c.$

I.F. for an equation of the form

$$x^ay^b\,(my\,dx + nx\,dy) + x^cy^d\,(py\,dx + qx\,dy) = 0$$

where a, b, c, d, m, n, p, q are all constant is x^hy^k, where h, k are so chosen that after multiplication by x^hy^k the equation becomes exact.

Example 11. *Solve* $(2x^2y^2 + y)\, dx + (3x - x^3y)\, dy = 0.$

Sol. The equation can be written as $2(x^2y^2\, dx - x^3ydy) + (y\, dx + 3xdy) = 0$

or $$x^2y\,(2ydx - xdy) + x^0y^0\,(y\,dx + 3xdy) = 0$$

which is of the form mentioned above. Therefore, it has an I.F. of the form x^hy^k.

Multiplying the given equation by x^hy^k, we have

$$(2x^{h+2}y^{k+2} + x^h\,y^{k+1})\,dx + (3x^{h+1}y^k - x^{h+3}y^{k+1})\,dy = 0$$

For this equation to be exact, we must have $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

i.e., $$2(k + 2)\,x^{h+2}y^{k+1} + (k + 1)\,x^hy^k = 3(h + 1)\,x^hy^k - (h + 3)\,x^{h+2}y^{k+1}$$

which holds when $2(k + 2) = -(h + 3)$ and $k + 1 = 3(h + 1)$

i.e., when $h + 2k + 7 = 0$ and $3h - k + 2 = 0$

Solving these equation, we have $h = -\frac{11}{7}, k = -\frac{19}{7}$

$\therefore$ $$\text{I.F.} = x^{-\frac{11}{7}} y^{-\frac{19}{7}}$$

Multiplying the given equation by $x^{-\frac{11}{7}} y^{-\frac{19}{7}}$, we have

$$\left(2x^{\frac{3}{7}} y^{-\frac{5}{7}} + x^{-\frac{11}{7}} y^{-\frac{12}{7}}\right) dx + \left(3x^{-\frac{4}{7}} y^{-\frac{19}{7}} - x^{\frac{10}{7}} y^{-\frac{12}{7}}\right) dy = 0$$

which is exact. The solution is $\displaystyle\int_{y \text{ constant}} \left(2x^{\frac{3}{7}} y^{-\frac{5}{7}} + x^{-\frac{11}{7}} y^{-\frac{12}{7}}\right) dx = c$

or $$\frac{7}{5} x^{\frac{10}{7}} y^{-\frac{5}{7}} - \frac{7}{4} x^{-\frac{4}{7}} y^{-\frac{12}{7}} = c \quad \text{or} \quad 4x^{\frac{10}{7}} y^{-\frac{5}{7}} - 5x^{-\frac{4}{7}} y^{-\frac{12}{7}} = C, \text{ where } C = \frac{20}{7} c.$$

Note. The values of h and k can also be determined from the relations

$$\frac{a+h+1}{m} = \frac{b+k+1}{n} \quad \text{and} \quad \frac{c+h+1}{p} = \frac{d+k+1}{q}.$$

Comparing the given equation

$$x^2y(2y\,dx - x\,dy) + x^0y^0(y\,dx + 3x\,dy) = 0$$

with $$x^ay^b\,(my\,dx + nx\,dy) + x^cy^d\,(py\,dx + qx\,dy) = 0$$

we have $$a = 2,\ b = 1,\ c = 0,\ d = 0$$

$$m = 2,\ n = -1,\ p = 1,\ q = 3$$

$\therefore$ $$\frac{a+h+1}{m} = \frac{b+k+1}{n} \Rightarrow \frac{2+h+1}{2} = \frac{1+k+1}{-1}$$

or $$3 + h = -4 - 2k \quad \text{or} \quad h + 2k + 7 = 0 \qquad \ldots(1)$$

Also, $$\frac{c+h+1}{p} = \frac{d+k+1}{q} \Rightarrow \frac{0+h+1}{1} = \frac{0+k+1}{3}$$

or $$3h - k + 2 = 0 \qquad \ldots(2)$$

Solving (1) and (2), we have $h = -\frac{11}{7},\ k = -\frac{19}{7}$.

EXERCISE E

Solve the following differential equations:

1. $(x^2y + y^4)dx + (2x^3 + 4xy^3)dy = 0$
2. $(y^2 + 2x^2y)dx + (2x^3 - xy)dy = 0$
3. $(2x^2y - 3y^4)dx + (3x^3 + 2xy^3)dy = 0$
4. $(y^3 - 2x^2y)dx + (2xy^2 - x^3)dy = 0$
5. $(2ydx + 3xdy) + 2xy(3ydx + 4xdy) = 0$
6. $x(3ydx + 2xdy) + 8y^4\,(ydx + 3xdy) = 0$
7. $(2y^2 - 4x^2y)\,dx + (4xy + 3x^3)\,dy = 0.$

Answers

1. $7x^{11/2}y^{11} + 11x^{7/2}y^{1/4} = c$ (I.F. $= x^{5/2}y^{10}$)
2. $6\sqrt{xy} - \left(\frac{y}{x}\right)^{3/2} = c$ (I.F. $= x^{-5/2}y^{-1/2}$)
3. $5x^{-36/13}\,y^{24/13} - 12x^{-10/13}\,y^{-15/13} = c$ (I.F. $= x^{-49/13}y^{-28/13}$)
4. $x^2y^4 - y^2x^4 = c$ (I.F. $= xy$)
5. $x^2y^3\,(1 + 2xy) = c$ (I.F. $= xy^2$)
6. $x^3y^2 + 4x^2y^6 = c$ (I.F. $= xy$)
7. $5x^{-\frac{2}{11}}y^{-\frac{4}{11}} + x^{\frac{20}{11}}y^{-\frac{5}{11}} = c$ (I.F. $= x^{-13/11}y^{-26/11}$)

UNIT III

3. LINEAR DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

STRUCTURE

Definition
To Solve the Equation + Py = Q, where P and Q are Functions of x only (Leibnitz's Equation)
Bernoulli's Equation (Equations Reducible to the Linear Form)
Differential equations of the first order and higher degree
Equations Solvable For p
Equations Solvable For y
Equations Solvable For x
Clairaut's Equation

DEFINITION

A differential equation is said to be linear if the dependent variable and its derivative occur only in the first degree and are not multiplied together.

Thus, the standard form of a linear differential equation of the first order is $\frac{dy}{dx}$ + Py = Q, where P and Q are functions of x or constants (*i.e.,* independent of y).

TO SOLVE THE EQUATION $\frac{dy}{dx}$ + Py = Q, WHERE P AND Q ARE FUNCTIONS OF x ONLY (Leibnitz's Equation)

The given equation is $\frac{dy}{dx} + \text{P}y = \text{Q}$

Multiplying throughout by $e^{\int \text{P}\,dx}$, we get

$$\frac{dy}{dx} \,.\, e^{\int \text{P}\,dx} + \text{P}y \,.\, e^{\int \text{P}\,dx} = \text{Q} \,.\, e^{\int \text{P}\,dx} \qquad ...(1)$$

NOTES

Now, $$\frac{d}{dx}[ye^{\int P\,dx}] = \frac{dy}{dx} \cdot e^{\int P\,dx} + y \cdot \frac{d}{dx}[e^{\int P\,dx}]$$

$$= \frac{dy}{dx} \cdot e^{\int P\,dx} + y \cdot e^{\int P\,dx} \cdot \frac{d}{dx}[\int P\,dx]$$

$$\left[\because \frac{d}{dx}\left\{e^{f(x)}\right\} = e^{f(x)} \cdot \frac{d}{dx}\{f(x)\}\right]$$

$$= \frac{dy}{dx} \cdot e^{\int P\,dx} + y \cdot e^{\int P\,dx} \cdot P = \frac{dy}{dx} \cdot e^{\int P\,dx} + Py \cdot e^{\int P\,dx}$$

$\therefore$ From (1), $$\frac{d}{dx}[y \cdot e^{\int P\,dx}] = Q \cdot e^{\int P\,dx}$$

Integrating both sides w.r.t. x, we have

$$y \cdot e^{\int P\,dx} = \int Q \cdot e^{\int P\,dx}\,dx + c$$

which is the required solution of the given linear differential equation.

Note 1. The factor $e^{\int P\,dx}$, on multiplying by which the LHS of the differential equation becomes the differential co-efficient of some function of x and y, is called an integrating factor of the differential equation and is shortly written as I.F.

Note 2. The solution of the linear equation $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x only, is

$$y(\text{I.F.}) = \int Q(\text{I.F.})\,dx + c$$

Note 3. Sometimes a differential equation becomes linear if we take y as the independent variable and x as dependent variable. In that case, the equation can be put in the form **$\frac{dx}{dy} + Px = Q$, where P and Q are functions of y (and not of x) or constants.**

I.F. (in this case) = $e^{\int P\,dy}$, and the solution is

$$x(\text{I.F.}) = \int Q.\ (\text{I.F.})\,dy + c.$$

Note 4. While evaluating the I.F., it is very useful to remember that

$$e^{\log f(x)} = f(x).$$

Thus, $$e^{\log x^2} = x^2.$$

Note 5. The co-efficient of $\frac{dy}{dx}$, if not unity, must be made unity by dividing throughout by it.

SOLVED EXAMPLES

Example 1. *Solve the following* :

(*i*) $(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2$ (*ii*) $\frac{dy}{dx} = y \tan x - 2 \sin x$.

Sol. (*i*) Given equation is $(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2$

Dividing throughout by $1 + x^2$, $\left(\text{to make the co-efficient of } \frac{dy}{dx} \text{ unity.}\right)$

$$\frac{dy}{dx} + \frac{2x}{1+x^2} \cdot y = \frac{4x^2}{1+x^2} \qquad \ldots(i)$$

It is of the form $\frac{dy}{dx} + Py = Q$

Here, $P = \frac{2x}{1+x^2}, Q = \frac{4x^2}{1+x^2}$

$\therefore$ $\text{I.F.} = e^{\int P\,dx} = e^{\int \frac{2x}{1+x^2}dx} = e^{\log(1+x^2)} = 1 + x^2$

Hence the solution is

$$y \,.\, (\text{I.F.}) = \int Q \,.\, (\text{I.F.})\, dx + c$$

or $$y(1 + x^2) = \int \frac{4x^2}{1+x^2} \,.\, (1 + x^2)\, dx + c$$

or $$y(1 + x^2) = \int 4x^2 dx + c$$

or $$y(1 + x^2) = \frac{4x^3}{3} + c.$$

(*ii*) Given equation is $\frac{dy}{dx} - (\tan x) \,.\, y = -\,2 \sin x$

It is of the form $\frac{dy}{dx} + Py = Q$

Here $P = -\tan x,\ Q = -\,2 \sin x$

$\therefore$ $\text{I.F.} = e^{\int P\,dx} = e^{-\int \tan x\, dx} = e^{-(-\log \cos x)}$

$= e^{\log \cos x} = \cos x$

Hence the solution is

$$y\,(\text{I.F.}) = \int Q \,.\, (\text{I.F.})\, dx + c$$

or $$y \cos x = \int -\,2 \sin x \cos x\, dx + c$$

$$= -\int \sin 2x\, dx + c = -\frac{-\cos 2x}{2} + c$$

or $$y \cos x = \tfrac{1}{2} \cos 2x + c.$$

Example 2. *Solve the following*:

(*i*) *sec x* $\frac{dy}{dx} = y + \sin x$ (*ii*) *x log x* $\frac{dy}{dx} + y = 2 \log x$

Sol. (*i*) Given equation is $\sec x \,.\, \frac{dy}{dx} - y = \sin x$

Dividing throughout by sec x, to make the co-efficient of $\frac{dy}{dx}$ unity,

$$\frac{dy}{dx} - (\cos x) \,.\, y = \sin x \cos x$$

It is of the form $\frac{dy}{dx} + Py = Q$

NOTES

Here, $\quad P = -\cos x, \quad Q = \sin x \cos x$

$\therefore \quad \text{I.F.} = e^{\int P\,dx} = e^{\int -\cos x\,dx} = e^{-\sin x}$

Hence the solution is

$$y\,.\,(\text{I.F.}) = \int Q\,.\,(\text{I.F.})\,dx + c$$

or $$y\,.\,e^{-\sin x} = \int \sin x \cos x\,.\,e^{-\sin x}\,dx + c = \int te^{-t}\,dt + c, \text{ where } t = \sin x$$

$$= t\,.\,\frac{e^{-t}}{-1} - \int 1.\frac{e^{-t}}{-1}\,dt + c = -\,te^{-t} - e^{-t} + c$$

$$= -\,e^{-t}(t+1) + c = -\,e^{-\sin x}(\sin x + 1) + c$$

or $$y = -\,(\sin x + 1) + c\,e^{\sin x}.$$

(*ii*) Given equation is $x \log x \dfrac{dy}{dx} + y = 2 \log x$

Dividing throughout by $x \log x$ to make the co-efficient of $\dfrac{dy}{dx}$ unity,

$$\frac{dy}{dx} + \frac{1}{x \log x}\,.\,y = \frac{2}{x}\,.$$

It is of the form $\quad \dfrac{dy}{dx} + Py = Q$

Here, $\quad P = \dfrac{1}{x \log x}, \quad Q = \dfrac{2}{x}$

$\therefore \quad \text{I.F.} = e^{\int P\,dx} = e^{\int \frac{1}{x \log x} dx} = e^{\int \frac{1/x}{\log x} dx} = e^{\log \log x} = \log x$

Hence the solution is

$$y\,.\,(\text{I.F.}) = \int Q.\,(\text{I.F.})\,dx + c$$

or $$y \log x = \int \frac{2}{x} \log x\,dx + c$$

or $$y \log x = 2 \int \frac{1}{x}\,.\,\log x\,dx + c$$

$$= 2\,.\,\frac{(\log x)^2}{2} + c \qquad \left|\; \because \int [f(x)^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}, n \neq -1 \right.$$

or $$y \log x = (\log x)^2 + c.$$

Example 3. *Solve:* $\quad x(x-1)\dfrac{dy}{dx} - (x-2)y = x^3(2x-1).$

Sol. Given equation is

$$x(x-1)\frac{dy}{dx} - (x-2)y = x^3(2x-1)$$

or $$\frac{dy}{dx} - \frac{x-2}{x(x-1)}\,y = \frac{x^2(2x-1)}{x-1}$$

It is of the form $\quad \dfrac{dy}{dx} + Py = Q$

Here, $\quad P = -\dfrac{x-2}{x(x-1)}, \; Q = \dfrac{x^2(2x-1)}{x-1}$

$\therefore \quad$ I.F. $= e^{\int P\,dx} = e^{-\int \frac{x-2}{x(x-1)}dx} = e^{-\int\left(\frac{2}{x}-\frac{1}{x-1}\right)dx}$

$= e^{-[2\log x - \log(x-1)]} = e^{-[\log x^2 - \log(x-1)]}$

$= e^{-\log\frac{x^2}{x-1}} = e^{\log\left(\frac{x^2}{x-1}\right)^{-1}} = \left(\dfrac{x^2}{x-1}\right)^{-1} = \dfrac{x-1}{x^2}$

$\therefore$ The solution is

$$y \cdot \frac{x-1}{x^2} = \int \frac{x^2(2x-1)}{x-1} \cdot \frac{x-1}{x^2}\,dx + c = \int (2x-1)\,dx + c = x^2 - x + c$$

or $\quad y(x-1) = x^2(x^2 - x + c).$

Example 4. *Solve:* $\quad x(1-x^2)\dfrac{dy}{dx} + (2x^2-1)y = x^3.$

Sol. Dividing by $x(1-x^2)$ to make the co-efficient of $\dfrac{dy}{dx}$ unity, the given equation becomes

$$\frac{dy}{dx} + \frac{2x^2-1}{x(1-x^2)}\,y = \frac{x^2}{1-x^2}$$

It is of the form $\dfrac{dy}{dx} + Py = Q$

Here $\quad P = \dfrac{2x^2-1}{x(1-x^2)}, \quad Q = \dfrac{x^2}{1-x^2}$

Now $\quad P = \dfrac{2x^2-1}{x(1-x)(1+x)} = -\dfrac{1}{x} + \dfrac{1}{2(1-x)} - \dfrac{1}{2(1+x)}$ $\quad$ [Partial fractions]

$\therefore \quad \int P\,dx = -\log x - \dfrac{1}{2}\log(1-x) - \dfrac{1}{2}\log(1+x)$

$= -\log[x(1-x)^{1/2}(1+x)^{1/2}]$

$= -\log[x\sqrt{1-x^2}\,] = \log(x\sqrt{1-x^2}\,)^{-1}$

$\therefore \quad$ I.F. $= e^{\int P\,dx} = e^{\log(x\sqrt{1-x^2})^{-1}} = \dfrac{1}{x\sqrt{1-x^2}}$

The solution is

$$y \cdot \frac{1}{x\sqrt{1-x^2}} = \int \frac{x^2}{1-x^2} \cdot \frac{1}{x\sqrt{1-x^2}}\,dx + c$$

$$= \int \frac{x}{(1-x^2)^{3/2}}\,dx + c = -\frac{1}{2}\int (1-x^2)^{-3/2} \cdot (-2x)\,dx + c$$

$$= -\frac{1}{2} \cdot \frac{(1-x^2)^{-1/2}}{-\frac{1}{2}} + c$$

NOTES

$$\Rightarrow \quad \frac{y}{x\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}} + c \quad \Rightarrow \quad y = x + cx\sqrt{1-x^2}$$

which is the required solution.

Example 5. *Solve:* $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1.$

Sol. The given equation is

$$\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$$

or $$\frac{dy}{dx} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}} \quad \text{or} \quad \frac{dy}{dx} + \frac{1}{\sqrt{x}}y = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

It is of the form $\frac{dy}{dx} + \text{P}y = \text{Q}$

Here $\text{P} = \frac{1}{\sqrt{x}}, \quad \text{Q} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

∴ $$\text{I.F.} = e^{\int \frac{1}{\sqrt{x}} dx} = e^{\int x^{-1/2} dx} = e^{2\sqrt{x}}$$

∴ Hence the solution is

$$y \cdot e^{2\sqrt{x}} = \int \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \cdot e^{2\sqrt{x}}\, dx + c$$

or $$y \cdot e^{2\sqrt{x}} = \int \frac{1}{\sqrt{x}}\, dx + c$$

or $$ye^{2\sqrt{x}} = 2\sqrt{x} + c \quad \text{or} \quad y = e^{-2\sqrt{x}}(2\sqrt{x} + c).$$

Equations of the Form $\frac{dx}{dy}$ + Px = Q where P and Q are functions of y only.

Example 6. *Solve the following:*

(*i*) $(1 + y^2) + (x - e^{tan^{-1} y})\frac{dy}{dx} = 0$ (*ii*) $(2x - 10y^3)\frac{dy}{dx} + y = 0.$

Sol. (*i*) The given equation is

$$(1 + y^2) + (x - e^{\tan^{-1} y})\frac{dy}{dx} = 0$$

or $$(1 + y^2)\frac{dx}{dy} + x - e^{\tan^{-1} y} = 0$$

or $$\frac{dx}{dy} + \frac{1}{1+y^2} \cdot x = \frac{e^{\tan^{-1} y}}{1+y^2}$$

NOTES

It is of the form $\frac{dx}{dy} + Px = Q$

$$\text{I.F.} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1} y}$$

$\therefore$ The solution is

$$x . e^{\tan^{-1} y} = \int \frac{e^{\tan^{-1} y}}{1+y^2} . e^{\tan^{-1} y}\, dy + c = \int e^t . e^t\, dt + c \quad \text{where } t = \tan^{-1} y$$

$$= \int e^{2t}\, dt + c = \tfrac{1}{2} e^{2t} + c$$

or $$x . e^{\tan^{-1} y} = \tfrac{1}{2} e^{2\tan^{-1} y} + c.$$

(*ii*) The given equation is $(2x - 10y^3)\frac{dy}{dx} + y = 0$

or $$y. \frac{dx}{dy} + 2x - 10y^3 = 0 \quad \text{or} \quad \frac{dx}{dy} + \frac{2}{y} . x = 10y^2$$

It is of the form $\frac{dx}{dy} + Px = Q$

$$\text{I.F.} = e^{\int P\,dx} = e^{\int \frac{2}{y}dy} = e^{2\log y} = e^{\log y^2} = y^2$$

$\therefore$ The solution is $$xy^2 = \int 10y^2 . y^2 dy + c = 10\int y^4\, dy + c$$

or $$xy^2 = \frac{10y^5}{5} + c = 2y^5 + c.$$

EXERCISE A

Solve the following differential equations:

1. $\frac{dy}{dx} + \frac{y}{x} = x^2$
2. $\frac{dy}{dx} + y \sec x = \tan x$
3. $\frac{dy}{dx} + y \tan x = \sec x$
4. $(1 + x^2)\frac{dy}{dx} + 2xy = \cos x$
5. $\frac{dy}{dx} = \frac{x+y+1}{x+1}$
6. $(x + 1)\frac{dy}{dx} - ny = e^x (x + 1)^{n+1}$
7. $\cos^2 x \frac{dy}{dx} + y = \tan x$
8. $(1 + x^2)\frac{dy}{dx} + y = \tan^{-1} x$
9. $\frac{dy}{dx} + \frac{2x}{x^2+1} . y = \frac{1}{(x^2+1)^2}$ given that $y = 0$ when $x = 1$
10. $\frac{dy}{dx} + 2y \tan x = \sin x$ given that $y = 0$ when $x = \frac{\pi}{3}$
11. $x\frac{dy}{dx} + 2y = x^2 \log x$
12. $\frac{dy}{dx} + y \cos x = \sin 2x$
13. $\frac{dy}{dx} = x(x^2 - 2y)$
14. $\sin x \frac{dy}{dx} + y \cos x = 2 \sin^2 x \cos x$

NOTES

15. $(1 - x^2)\dfrac{dy}{dx} + xy = ax$

16. $x(x-1)\dfrac{dy}{dx} - y = x^2(x-1)^2$

17. $y\,dx - x\,dy + \log x\,dx = 0$

18. $\dfrac{dy}{dx} + 2y\cot x = 3x^2\operatorname{cosec}^2 x$

19. $\sin 2x\dfrac{dy}{dx} = y + \tan x$

20. $(x + 2y^3)\dfrac{dy}{dx} = y$

21. $(1 + y^2)\,dx = (\tan^{-1} y - x)\,dy$

22. $e^y\,dx + (1 + xe^y)\,dy = 0$

23. $\dfrac{dx}{dy} + 2x = 6e^y$

24. $2y' + 4y = x^2 - x$

25. $y' - 2y = \cos 3x$

26. $\dfrac{dy}{dx} + y\cot x = 2x + x^2\cot x$

27. $y' + y = \dfrac{1 + x\log x}{x}$

28. $xy' - y = (x-1)\,e^x$

Answers

1. $xy = \dfrac{1}{4}x^4 + c$

2. $y(\sec x + \tan x) = \sec x + \tan x - x + c$

3. $y = \sin x + c\cos x$

4. $y(1 + x^2) = \sin x + c$

5. $\dfrac{y}{x+1} = \log(x+1) + c$

6. $y = (x+1)^n(e^x + c)$

7. $y = \tan x - 1 + ce^{-\tan x}$

8. $y = \tan^{-1} x - 1 + ce^{-\tan x}$

9. $y(x^2 + 1) = \tan^{-1} x - \dfrac{\pi}{4}$

10. $y = \cos x - 2\cos^2 x$

11. $x^2y = \dfrac{x^4}{4}\log x - \dfrac{x^4}{16} + c$

12. $y = 2(\sin x - 1) + ce^{-\sin x}$

13. $y = \dfrac{1}{2}(x^2 - 1) + ce^{-x^2}$

14. $y\sin x = \dfrac{2}{3}\sin^3 x + c$

15. $y = a + c\sqrt{1 - x^2}$

16. $y = \left(1 - \dfrac{1}{x}\right)\left(\dfrac{x^3}{3} + c\right)$

17. $y + 1 + \log x = cx$

18. $y\sin^2 x = x^3 + c$

19. $y = \tan x + c\sqrt{\tan x}$

20. $x = y^3 + cy$

21. $x = \tan^{-1} y - 1 + ce^{-\tan^{-1} y}$

22. $xe^y + y = c$

23. $x = 2e^y + ce^{-2y}$

24. $y = \dfrac{1}{4}(x-1)^2 + ce^{-2x}$

25. $y = \dfrac{1}{13}(3\sin 3x - 2\cos 3x) + ce^{2x}$

26. $y\sin x = x^2\sin x + c$

27. $y = \log x + ce^{-x}$

28. $y = e^x + cx$

BERNOULLI'S EQUATION (EQUATIONS REDUCIBLE TO THE LINEAR FORM)

To solve the equation $\dfrac{dy}{dx} + Py = Qy^n$, where P and Q are functions of x only

The given equation is $\dfrac{dy}{dx} + Py = Qy^n$ ...(*i*)

Dividing both sides of (i) by y^n, **to make the RHS a function of x only.**

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q \qquad ...(ii)$$

Put $y^{1-n} = z$, then

$$(1-n)\,.\,y^{-n}\frac{dy}{dx} = \frac{dz}{dx} \qquad \text{or} \qquad y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\,.\,\frac{dz}{dx}$$

$\therefore$ (ii) becomes $\frac{1}{1-n}\,.\,\frac{dz}{dx} + Pz = Q$

or $$\frac{dz}{dx} + (1-n)\,.\,Pz = (1-n)\,Q.$$

which is a linear equation in z and can be solved.

In the solution, putting $z = y^{1-n}$, we get the required solution.

SOLVED EXAMPLES

Example 7. *Solve:* $2\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$.

Sol. The given equation is $2\,.\,\frac{dy}{dx} - \frac{y}{x} = \frac{y^2}{x^2}$.

Dividing throughout by y^2

$$2y^{-2}\frac{dy}{dx} - \frac{1}{x}\,.\,y^{-1} = \frac{1}{x^2} \qquad ...(i)$$

Put $y^{-1} = z$, then $-y^{-2}\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore$ (i) becomes

$$-2\frac{dz}{dx} - \frac{1}{x}z = \frac{1}{x^2} \qquad \text{or} \qquad \frac{dz}{dx} + \frac{1}{2x}z = -\frac{1}{2x^2}$$

which is linear in z.

$$P = \frac{1}{2x},\ Q = -\frac{1}{2x^2}$$

$$\text{I.F.} = e^{\int \frac{1}{2x}dx} = e^{\frac{1}{2}\log x} = e^{\log\sqrt{x}} = \sqrt{x}$$

$\therefore$ The solution is $z\,.\,\sqrt{x} = \int -\frac{1}{2x^2}\sqrt{x}\,dx + c$

or $$y^{-1}\sqrt{x} = -\frac{1}{2}\int x^{-3/2}\,dx + c \qquad \text{or} \qquad \frac{\sqrt{x}}{y} = \frac{1}{\sqrt{x}} + c$$

or $$x = y(1 + c\sqrt{x}).$$

Example 8. *Solve the following* :

(i) $\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x^2}$ (ii) $\frac{dy}{dx} + \frac{x}{1-x^2}y = x\sqrt{y}$.

NOTES

Sol. (*i*) The given equation is $\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x^2}$

Dividing throughout by e^y

$$e^{-y}\frac{dy}{dx} + e^{-y}\frac{dy}{dx} + \frac{1}{x} = \frac{1}{x^2} \quad \text{...}(i)$$

Put $e^{-y} = z$, then $\quad -e^{-y}\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore$ (*i*) becomes

$$-\frac{dz}{dx} + z \cdot \frac{1}{x} = \frac{1}{x^2} \quad \text{or} \quad \frac{dz}{dx} - \frac{1}{x} \cdot z = -\frac{1}{x^2}$$

which is linear in z. $\quad P = -\frac{1}{x}$, $Q = -\frac{1}{x^2}$

$$\text{I.F.} = e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log \frac{1}{x}} = \frac{1}{x}$$

$\therefore$ The solution is $\quad z \cdot \frac{1}{x} = \int -\frac{1}{x^2} \cdot \frac{1}{x}\, dx + c$

or $\quad e^{-y} \cdot \frac{1}{x} = -\int \frac{1}{x^3}\, dx + c \quad$ or $\quad e^{-y} \cdot \frac{1}{x} = \frac{1}{2x^2} + c$

or $\quad 2x = e^y + 2cx^2e^y$.

(*ii*) The given equation is $\frac{dy}{dx} + \frac{x}{1-x^2}\, y = x\sqrt{y}$

Dividing throughout by $\sqrt{y}$,

$$y^{1/2} \cdot \frac{dy}{dx} + \frac{x}{1-x^2}\, y^{1/2} = x \quad \text{...}(i)$$

Put $y^{1/2} = z$; then $\quad \frac{1}{2}y^{-1/2} \cdot \frac{dy}{dx} = \frac{dz}{dx}$

$\therefore$ (*i*) becomes

$$2 \cdot \frac{dz}{dx} + \frac{x}{1-x^2} \cdot z = x \quad \text{or} \quad \frac{dz}{dx} + \frac{x}{2(1-x^2)} \cdot z = \frac{x}{2}$$

which is linear in z. $\quad P = \frac{x}{2(1-x^2)}$, $Q = \frac{x}{2}$

$$\therefore \quad \text{I.F.} = e^{\int \frac{x}{2(1-x^2)}dx} = e^{-\frac{1}{4}\int \frac{-2x}{1-x^2}dx} \quad | \textbf{ Note}$$

$$= e^{-\frac{1}{4}\log(1-x^2)} = e^{\log(1-x^2)^{-1/4}} = (1-x^2)^{-1/4}$$

$\therefore$ The solution is $\quad z \cdot (1-x^2)^{-1/4} = \int \frac{x}{2}(1-x^2)^{-1/4}\, dx + c$

or $\quad \sqrt{y} \cdot (1-x^2)^{-1/4} = -\frac{1}{4}\int -2x(1-x^2)^{-1/4}\, dx + c$

or $\quad \sqrt{y} \cdot (1-x^2)^{-1/4} = -\frac{1}{4} \cdot \frac{(1-x^2)^{3/4}}{\frac{3}{4}} + c$

or $\quad \sqrt{y} = -\frac{1}{3}(1-x^2) + c(1-x^2)^{1/4}$.

Example 9. *Solve:* $(x^2y^3 + xy)\, dy = dx.$

Sol. The given equation is $(x^2y^3 + xy)dy = dx$

or $$\frac{dx}{dy} = x^2y^3 + xy$$

or $$\frac{dx}{dy} - xy = x^2y^3 \qquad \left| \text{Form } \frac{dx}{dy} + \text{P}x = \text{Q}x^n \right.$$

Dividing throughout by x^2

$$x^{-2}\frac{dx}{dy} - x^{-1}\, y = y^3 \qquad \ldots(i)$$

Put $x^{-1} = z$, then $\quad -x^{-2}\dfrac{dx}{dy} = \dfrac{dz}{dy}$

$\therefore$ (*i*) becomes

$$-\frac{dz}{dy} - zy = y^3 \quad \text{or} \quad \frac{dz}{dy} + y.z = -y^3$$

which is linear in z. $\quad \text{P} = y,\ \text{Q} = -y^3$

$$\text{I.F.} = e^{\int y\, dy} = e^{y^2/2}$$

$\therefore$ The solution is $\quad z \,.\, e^{1/2y^2} = \displaystyle\int -y^3 \,.\, e^{1/2y^2}\, dy + c$

or $$x^{-1} \,.\, e^{1/2y^2} = -\int y^2 \,.\, y \,.\, e^{1/2y^2}\, dy + c$$

$$= -\int 2t\, e^t\, dt + c, \quad \text{where } t = \tfrac{1}{2}\, y^2$$

or $$x^{-1} \,.\, e^{1/2y^2} = -2e^t\, (t-1) + c$$

or $$x^{-1} \,.\, e^{1/2y^2} = -2e^{1/2y^2}\, (\tfrac{1}{2}\, y^2 - 1) + c$$

or $$x^{-1} = -y^2 + 2 + ce^{-1/2y^2}.$$

Example 10. *Solve the following*:

(*i*) $(x + 1)\dfrac{dy}{dx} + 1 = 2e^{-y}$ $\qquad$ (*ii*) $\dfrac{dy}{dx} = e^{x-y}\, (e^x - e^y)$

Sol. (*i*) The given equation is $(x + 1)\dfrac{dy}{dx} + 1 = 2e^{-y}$

or $$\frac{dy}{dx} + \frac{1}{x+1} = \frac{2e^{-y}}{x+1}$$

or $$e^y \,.\, \frac{dy}{dx} + \frac{1}{x+1} \,.\, e^y = \frac{2}{x+1} \qquad \ldots(i)$$

Put $e^y = z$, then $\quad e^y \,.\, \dfrac{dy}{dx} = \dfrac{dz}{dx}$

$\therefore$ From (*i*), $\quad \dfrac{dz}{dx} + \dfrac{1}{x+1} \,.\, z = \dfrac{2}{x+1}$

which is linear in z. $\quad \text{P} = \dfrac{1}{x+1},\ \text{Q} = \dfrac{2}{x+1}$

$$\text{I.F.} = e^{\int \frac{1}{x+1} dx} = e^{\log (x+1)} = x + 1$$

$\therefore$ The solution is $\quad z\, (x + 1) = \displaystyle\int \frac{2}{x+1} \,.\, (x + 1)\, dx + c$

or $$c^y \,.\, (x + 1) = 2x + c.$$

NOTES

(*ii*) The given equation is

$$\frac{dy}{dx} = e^{x-y}\,(e^x - e^y) \quad \text{or} \quad \frac{dy}{dx} = e^{2x}\,.\,e^{-y} - e^x$$

or $$\frac{dy}{dx} + e^x = e^{2x}\,.\,e^{-y} \quad \text{or} \quad e^y\,.\,\frac{dy}{dx} + e^x\,.\,e^y = e^{2x} \qquad ...(i)$$

Put $e^y = z$, then $e^y\,.\,\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore$ (*i*) becomes $\frac{dz}{dx} + e^x\,.\,z = e^{2x}$

which is linear in *z*. $\text{P} = e^x,\ \text{Q} = e^{2x}$

$$\text{I.F.} = e^{\int e^x\,dx} = e^{e^x}$$

$\therefore$ The solution is $z\,.\,e^{e^x} = \int e^{2x}\,.\,e^{e^x}\,dx + c$

or $$e^y\,.\,e^{e^x} = \int e^x\,.\,e^x\,.\,e^{e^x}\,dx + c$$

$$= \int t\,e^t\,dt + c, \quad \text{where } t = e^x$$

$$= e^t\,(t - 1) + c$$

or $$e^y\,.\,e^{e^x} = e^{e^x}\,(e^x - 1) + c \quad \text{or} \quad e^y = e^x - 1 + c\,e^{-e^x}.$$

Example 11. *Solve:* $\frac{dy}{dx} + (2x\tan^{-1} y - x^3)(1 + y^2) = 0.$

Sol. The given equation is

$$\frac{dy}{dx} + (2x\tan^{-1} y - x^3)(1 + y^2) = 0$$

or $$\frac{1}{1+y^2}\,.\,\frac{dy}{dx} + 2x\tan^{-1} y - x^3 = 0$$

or $$\frac{1}{1+y^2}\,.\,\frac{dy}{dx} + 2x\tan^{-1} y = x^3 \qquad ...(i)$$

Put $\tan^{-1} y = z$, then $\frac{1}{1+y^2}\,.\,\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore$ From (*i*), $\frac{dz}{dx} + 2xz = x^3$

which is linear in *z*. $\text{P} = 2x,\ \text{Q} = x^3$

$$\text{I.F.} = e^{\int 2x\,dx} = e^{x^2}$$

$\therefore$ The solution is $z\,.\,e^{x^2} = \int x^3\,.\,e^{x^2}\,dx + c$

or $$\tan^{-1} y\,.\,e^{x^2} = \tfrac{1}{2}\int 2x\,.\,x^2\,e^{x^2}\,dx + c$$

$$= \tfrac{1}{2}\int t\,e^t\,dt + c, \quad \text{where } t = x^2$$

$$= \tfrac{1}{2}\,e^t\,(t - 1) + c$$

or $$\tan^{-1} y\,.\,e^{x^2} = \tfrac{1}{2}\,e^{x^2}\,(x^2 - 1) + c$$

or $$\tan^{-1} y = \tfrac{1}{2}\,(x^2 - 1) + ce^{-x^2}.$$

Example 12. *Solve the following differential equations*:

(*i*) $(x^3y^2 + xy)\,dx = dy$ (*ii*) $\frac{dy}{dx} + \frac{y}{x}\log y = \frac{y}{x^2}\,(\log y)^2$

Sol. (*i*) The given equation is $(x^3y^2 + xy)\, dx = dy$

or $$\frac{dy}{dx} = x^3y^2 + xy \quad \text{or} \quad \frac{dy}{dx} - xy = x^3y^2$$

Dividing both sides by y^2, $$y^{-2}\frac{dy}{dx} - xy^{-1} = x^3 \quad \ldots(i)$$

Put $y^{-1} = z$, then $$-y^{-2}\frac{dy}{dx} = \frac{dz}{dx}$$

$\therefore$ (*i*) becomes $$\frac{dz}{dx} - xz = x^3 \quad \text{or} \quad \frac{dz}{dx} + xz = -x^3$$

which is linear in z. $\quad \mathrm{P} = x, \quad \mathrm{Q} = -x^3$

$$\text{I.F.} = e^{\int x\,dx} = e^{\frac{x^2}{2}}$$

$\therefore$ The solution is $$z\,.\,e^{\frac{x^2}{2}} = \int -x^3\,.\,e^{\frac{x^2}{2}}\,dx + c = -\int x^2\,.\,xe^{\frac{x^2}{2}}\,dx + c$$

$$= -\int 2te^t\,dt + c, \quad \text{where } t = \frac{x^2}{2}$$

$$= -\int 2te^t dt + c = -2e^t\,(t-1) + c$$

$$y^{-1}\,.\,e^{\frac{x^2}{2}} = -2e^{\frac{x^2}{2}}\left(\frac{x^2}{2} - 1\right) + c$$

or $$y^{-1} = -x^2 + 2 + ce^{-\frac{x^2}{2}}.$$

(*ii*) The given equation is $\dfrac{dy}{dx} + \dfrac{y}{x}\log y = \dfrac{y}{x^2}(\log y)^2$

Dividing both sides by $y\,(\log y)^2$, we get

$$\frac{1}{y(\log y)^2}\,.\,\frac{dy}{dx} + \frac{1}{\log y}\,.\,\frac{1}{x} = \frac{1}{x^2} \quad \ldots(i)$$

Put $\dfrac{1}{\log y} = (\log y)^{-1} = z$, then $-(\log y)^{-2}\,.\,\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{dz}{dx}$

or $$\frac{1}{y(\log y)^2}\,.\,\frac{dy}{dx} = -\frac{dz}{dx}$$

$\therefore$ From (*i*), $$-\frac{dz}{dx} + z\,.\,\frac{1}{x} = \frac{1}{x^2} \quad \text{or} \quad \frac{dz}{dx} - \frac{1}{x}z = -\frac{1}{x^2}$$

which is linear in z. $\quad \mathrm{P} = -\dfrac{1}{x}, \ \mathrm{Q} = -\dfrac{1}{x^2}.$

$$\text{I.F.} = e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$$

$\therefore$ The solution is $$z\,.\,\frac{1}{x} = \int -\frac{1}{x^2}\,.\,\frac{1}{x}\,dx + c$$

or $$z\,.\,\frac{1}{x} = -\int x^{-3}\,dx + c = -\frac{x^{-2}}{-2} + c$$

or $$\frac{1}{\log y}\,.\,\frac{1}{x} = \frac{1}{2x^2} + c \quad \text{or} \quad \frac{1}{\log y} = \frac{1}{2x} + cx.$$

NOTES

Example 13. *Show how to solve an equation of the form*

$$f'(y)\frac{dy}{dx} + Pf(y) = Q \quad \text{where P, Q are functions of } x \text{ only.}$$

Sol. (*a*) The given equation is

$$f'(y)\frac{dy}{dx} + \text{P}f(y) = \text{Q} \qquad \ldots(i)$$

where P, Q are functions of x only.

Put $f(y) = z$, then $\quad f'(y)\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore$ (*i*) becomes $\quad \frac{dz}{dx} + \text{P}z = \text{Q}$

which is linear in z and can be solved.

$$\text{I.F.} = e^{\int \text{P}\,dx} \text{ and the solution is}$$

$$z(\text{I.F.}) = \int \text{Q} \,.\, (\text{I.F.})\, dx + c$$

or $\quad f(y)\,.\,(\text{I.F.}) = \int \text{Q}\,.\,(\text{I.F.})\,dx + c$

Example 14. *Solve the following differential equations*:

(*i*) $(x+1)\frac{dy}{dx} + 1 = e^{x-y}$ $\qquad$ (*ii*) $\frac{dy}{dx} = y \tan x - y^2 \sec x.$

Sol. (*i*) The given equation is

$$(x+1)\frac{dy}{dx} + 1 = \frac{e^x}{e^y} \quad \text{or} \quad e^y\frac{dy}{dx} + \frac{e^y}{x+1} = \frac{e^x}{x+1} \qquad \ldots(i)$$

Putting $e^y = z$ so that $\quad e^y\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore$ (*i*) becomes $\quad \frac{dz}{dx} + \frac{z}{x+1} = \frac{e^x}{x+1}$

which is linear in z with $\quad \text{P} = \frac{1}{x+1}, \ \text{Q} = \frac{e^x}{x+1}$

$$\text{I.F.} = e^{\int \text{P}\,dx} = e^{\int \frac{1}{x+1}dx} = e^{\log(x+1)} = x + 1$$

$\therefore$ The solution is $\quad z(x+1) = \int \frac{e^x}{x+1}\,.\,(x+1)\,dx + c \quad$ or $\quad e^y(x+1) = e^x + c.$

(*ii*) The given equation is

$$\frac{dy}{dx} - y\tan x = -y^2 \sec x$$

or $\quad -\frac{1}{y^2}\,.\,\frac{dy}{dx} + \frac{1}{y}\tan x = \sec x \qquad \ldots(1)$

Putting $\frac{1}{y} = z$ so that $-\frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore$ Equation (1) becomes $\dfrac{dz}{dx} + z \tan x = \sec x$

which is linear in z with $\quad P = \tan x, \quad Q = \sec x$

$$\text{I.F.} = e^{\int P\,dx} = e^{\int \tan x\,dx} = e^{\log \sec x} = \sec x$$

$\therefore$ The solution is $\quad z \,.\, \sec x = \int \sec x \,.\, \sec x\, dx + c$

or $\quad \dfrac{1}{y} \sec x = \tan x + c \quad$ or $\quad \dfrac{1}{y} = \sin x + c \cos x.$

EXERCISE B

Solve the following differential equations:

1. $\dfrac{dy}{dx} + \dfrac{y}{x} = y^2$

2. $y' + y = y^2$

3. $\dfrac{dy}{dx} = x^3y^3 - xy$

4. $3\dfrac{dy}{dx} + \dfrac{2}{x+1} y = \dfrac{x^3}{y^2}$

5. $\dfrac{dy}{dx} + \dfrac{y}{x} = x^2y^6$

6. $x\dfrac{dy}{dx} + y = x^3y^4$

7. $\dfrac{dy}{dx} - 2y \tan x = y^2 \tan^2 x$

8. $(y \log x - 1)y\, dx = x\, dy$

9. $\dfrac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$

10. $\dfrac{dy}{dx} - \dfrac{\tan y}{1+x} = (1 + x)\, e^x \sec y$

11. $(x - y^2)\, dx + 2xy\, dy = 0$

12. $\cos x\, dy = y(\sin x - y)\, dx$

13. $xy - \dfrac{dy}{dx} = y^3 e^{-x^2}$

14. $(xy^2 - e^{1/x^3})\, dx - x^2y\, dy = 0$

15. $e^y \left(\dfrac{dy}{dx} + 1\right) = e^x$

16. $(xy - 2x \log x)\, dy = 2y\, dx$

17. $\dfrac{dy}{dx} + \dfrac{y}{x} = y^2 \log x$

18. $y\,(2xy + e^x)\, dx = e^x y$

19. $x\dfrac{dy}{dx} + y = y^2x^3 \cos x$

20. $\sin y \dfrac{dy}{dx} = \cos y\,(1 - x \cos y)$

Answers

1. $\dfrac{1}{xy} + \log x = c$

2. $y = \dfrac{1}{1 + ce^x}$

3. $y^{-2} = x^2 + 1 + ce^{x^2}$

4. $y^2(x + 1)^2 = \dfrac{x^6}{6} + \dfrac{2x^5}{5} + \dfrac{x^4}{4} + c$

5. $\dfrac{1}{y^5} = \dfrac{5}{2}x^3 + cx^5$

6. $\dfrac{1}{y^3} = -3x^3 \log x + cx^3$

7. $\dfrac{1}{y}\sec^2 x = -\dfrac{\tan^3 x}{3} + c$

8. $\dfrac{1}{y} = \log x + 1 + cx$

9. $\tan y = \dfrac{1}{2}(x^2 - 1) + ce^{-x^3}$

10. $\sin y = (1 + x)(e^x + c)$

11. $y^2 = x\,(c - \log x)$

12. $\dfrac{1}{y} = \sin x + c \cos x$

NOTES

13. $y^{-2} \cdot e^{x^2} = 2x + c$

14. $3y^2 = 2x^2 e^{\frac{1}{x^3}} + cx^2$

15. $e^{x+y} = \frac{1}{2} e^{2x} + c$

16. $y \log x = \frac{y^2}{4} + c$

17. $\frac{1}{y} = -\frac{1}{2}(\log x)^2 + cx$

18. $e^x = y(c - x^2)$

19. $\frac{1}{xy} = -x \sin x - \cos x + c$

20. $\sec y = x + 1 + ce^x$

DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AND HIGHER DEGREE

So far, we have discussed differential equations of the first order and first degree. Now we shall study differential equations of the first order and degree higher than the first. For convenience, we denote $\frac{dy}{dx}$ by p.

A differential equation of the first order and n^{th} degree is of the form

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \ldots\ldots + P_n = 0 \qquad \ldots(1)$$

where $P_1, P_2, \ldots\ldots, P_n$ are functions of x and y.

Since it is a differential equation of the first order, its general solution will contain only one arbitrary constant.

In the various cases which follow, the problem is reduced to that of solving one or more equations of the first order and first degree.

EQUATIONS SOLVABLE FOR *p*

Resolving the left hand side of (1) into n linear factors, we have

$$[p - f_1(x, y)]\ [p - f_2(x, y)], \ldots\ldots, [p - f_n(x, y)] = 0$$

which is equivalent to $p - f_1(x, y) = 0,\ p - f_2(x, y) = 0, \ldots\ldots, p - f_n(x, y) = 0$

Each of these equations is of the first order and first degree and can be solved by the methods already discussed.

If the solutions of the above n component equations are

$$F_1(x, y, c) = 0,\ F_2(x, y, c) = 0, \ldots\ldots, F_n(x, y, c) = 0$$

then the general solution of (1) is given by

$$F_1(x, y, c) \cdot F_2(x, y, c) \ldots\ldots F_n(x, y, c) = 0.$$

SOLVED EXAMPLES

Example 15. *Solve:* $x^2 \left(\frac{dy}{dx}\right)^2 + xy \frac{dy}{dx} - 6y^2 = 0.$

Sol. The given equation is $x^2p^2 + xyp - 6y^2 = 0$ where $p = \frac{dy}{dx}$

Factorising $(xp + 3y)(xp - 2y) = 0$

$\Rightarrow$ $xp + 3y = 0$ or $xp - 2y = 0$

Now, $xp + 3y = 0$

$\Rightarrow$ $x\frac{dy}{dx} + 3y = 0$ or $\frac{dy}{y} + 3\frac{dx}{x} = 0$

Integrating, $\log y + 3 \log x = \log c$ or $x^3y = c$

Also, $xp - 2y = 0$

$\Rightarrow$ $x\frac{dy}{dx} - 2y = 0$ or $\frac{dy}{y} - 2\frac{dx}{x} = 0$

Integrating, $\log y - 2 \log x = \log c$ or $\frac{y}{x^2} = c$ or $y = cx^2$

$\therefore$ The general solution of the given equation is $(x^3y - c)(y - cx^2) = 0$.

Example 16. *Solve* $xyp^2 + p(3x^2 - 2y^2) - 6xy = 0$.

Sol. Solving the given equation for p, we have

$$p = \frac{-(3x^2 - 2y^2) \pm \sqrt{(3x^2 - 2y^2)^2 + 24x^2y^2}}{2xy}$$

$$= \frac{(2y^2 - 3x^2) \pm (3x^2 + 2y^2)}{2xy} = \frac{2y}{x} \quad \text{or} \quad -\frac{3x}{y}$$

Now, $p = \frac{2y}{x} \Rightarrow \frac{dy}{dx} = \frac{2y}{x}$ or $\frac{dy}{y} - \frac{2dx}{x} = 0$

Integrating, $\log y - 2 \log x = \log c$ or $\frac{y}{x^2} = c$ or $y = cx^2$

Also, $p = -\frac{3x}{y} \Rightarrow \frac{dy}{dx} = -\frac{3x}{y}$ or $ydy + 3xdx = 0$

Integrating, $\frac{y^2}{2} + \frac{3x^2}{2} = C$ or $y^2 + 3x^2 = c$

$\therefore$ The general solution of the given equation is $(y - cx^2)(y^2 + 3x^2 - c) = 0$.

Example 17. *Solve* $p^2 + 2py \cot x = y^2$.

Sol. The given equation can be written as $(p + y \cot x)^2 = y^2 (1 + \cot^2 x)$

or $p + y \cot x = \pm y \operatorname{cosec} x$

$\therefore$ The component equations are

$$p = y(-\cot x + \operatorname{cosec} x) \qquad \ldots(1)$$

and

$$p = y(-\cot x - \operatorname{cosec} x) \qquad \ldots(2)$$

From (1), $\frac{dy}{dx} = y(-\cot x + \operatorname{cosec} x)$

or $\frac{dy}{y} = (-\cot x + \operatorname{cosec} x)\, dx$

Integrating, $\log y = -\log \sin x + \log \tan \frac{x}{2} + \log c = \log \frac{c \tan \frac{x}{2}}{\sin x}$

NOTES

or $$y = \frac{c \tan \frac{x}{2}}{2 \sin \frac{x}{2} \cos \frac{x}{2}} = \frac{c}{2 \cos^2 \frac{x}{2}} = \frac{c}{1 + \cos x}$$

or $$y(1 + \cos x) = c$$

From (2), $$\frac{dy}{dx} = y(-\cot x - \text{cosec } x)$$

or $$\frac{dy}{y} = (-\cot x - \text{cosec } x)\, dx$$

Integrating, $$\log y = -\log \sin x - \log \tan \frac{x}{2} + \log c = \log \frac{c}{\sin x \tan \frac{x}{2}}$$

or $$y = \frac{c}{2 \sin^2 \frac{x}{2}} = \frac{c}{1 - \cos x} \quad \text{or} \quad y(1 - \cos x) = c$$

$\therefore$ The general solution of the given equation is

$$[y(1 + \cos x) - c]\, [y(1 - \cos x) - c] = 0.$$

EXERCISE C

Solve the following equations:

1. $p^2 - 7p + 12 = 0$

2. $xy \left(\frac{dy}{dx}\right)^2 - (x^2 + y^2) \frac{dy}{dx} + xy = 0$

3. $yp^2 + (x - y)p - x = 0$

4. $x^2 \left(\frac{dy}{dx}\right)^2 + 3xy \frac{dy}{dx} + 2y^2 = 0$

5. $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$

6. $p^2 - 2p \sinh x - 1 = 0$

7. $p\,(p + y) = x\,(x + y)$

8. $4y^2p^2 + 2pxy\,(3x + 1) + 3x^3 = 0.$

Answers

1. $(y - 4x - c)\,(y - 3x - c) = c$ $= 0$

2. $(y^2 - x^2 - c)\,(y - cx) = 0$

3. $(y - x - c)\,(x^2 + y^2 - c)$

4. $(xy - c)(x^2y - c) = 0$

5. $(xy - c)(x^2 - y^2 - c) = 0$

6. $(y - e^x - c)\,(y - e^{-x} - c)$ $= 0$

7. $(y - \frac{1}{2} x^2 + c)\,(y + x + ce^{-x} - 1) = 0$

8. $(y^2 + x^3 - c)(y^2 + \frac{1}{2} x^2 - c) = 0.$

EQUATIONS SOLVABLE FOR *y*

If the equation is solvable for y, we can express y explicitly in terms of x and p. Thus, the equations of this type can be put as $y = f\,(x, p)$ ...(1)

Differentiating (1) w.r.t. x, we get $\frac{dy}{dx} = p = F\left(x, p, \frac{dp}{dx}\right)$ ...(2)

Equation (2) is a differential equation of first order in p and x.

Suppose the solution of (2) is $\phi(x, p, c) = 0$ ...(3)

Now elimination of p from (1) and (3) gives the required solution.

If p cannot be easily eliminated, then we solve equations (1) and (3) for x and y to get

$$x = \phi_1 (p, c), y = \phi_2 (p, c)$$

These two relations together constitute the solution of the given equation with p as parameter.

SOLVED EXAMPLES

Example 18. *Solve* $y + px = x^4p^2$.

Sol. Given equation is $y = -px + x^4p^2$ ...(1)

Differentiating both sides w.r.t. x,

$$\frac{dy}{dx} = p = -p - x\frac{dp}{dx} + 4x^3p^2 + 2x^4p\frac{dp}{dx}$$

or $$2p + x\frac{dp}{dx} - 2px^3\left(2p + x\frac{dp}{dx}\right) = 0$$

or $$\left(2p + x\frac{dp}{dx}\right)(1 - 2px^3) = 0$$

Discarding the factor $(1 - 2px^3)$, which does not involve $\frac{dp}{dx}$, we have

$$2p + x\frac{dp}{dx} = 0 \quad \text{or} \quad \frac{dp}{p} + 2\frac{dx}{x} = 0$$

Integrating, $\log p + 2 \log x = \log c$

or $\log px^2 = \log c$

or $px^2 = c \Rightarrow p = \frac{c}{x^2}$.

Putting this value of p in (1), we have $y = -\frac{c}{x} + c^2$.

Example 19. *Solve* $y = 2px - p^2$.

Sol. The given equation is $y = 2px - p^2$ ...(1)

Differentiating both sides w.r.t. x, $\frac{dy}{dx} = p = 2p + 2x\frac{dp}{dx} - 2p\frac{dp}{dx}$

or $$p + (2x - 2p)\frac{dp}{dx} = 0$$

or $$p\frac{dx}{dp} + 2x - 2p = 0$$

or $$\frac{dx}{dp} + \frac{2}{p}x = 2 \quad \text{...(2)}$$

which is a linear equation.

$$\text{I.F.} = e^{\int \frac{2}{p}dp} = e^{2 \log p} = p^2$$

$\therefore$ The solution of (2) is $x \text{ I.F.} = \int 2 \text{I.F.} dp + c$

or $$xp^2 = \int 2p^2 dp + c$$

or $$xp^2 = \frac{2}{3}p^3 + c \quad \text{or} \quad x = \frac{2}{3}p + cp^{-2} \qquad ...(3)$$

Putting this value of x in (1), we have $$y = 2p\left(\frac{2}{3}p + cp^{-2}\right) - p^2$$

NOTES

or $$y = \frac{1}{3}p^2 + 2cp^{-1}. \qquad ...(4)$$

Equations (3) and (4) together constitute the general solution of (1).

EXERCISE D

Solve the following equations:

1. $xp^2 - 2yp + ax = 0$ **2.** $y - 2px = \tan^{-1}(xp^2)$ **3.** $16x^2 + 2p^2y - p^3x = 0$

4. $y = x + 2\tan^{-1} p$ **5.** $y = 3x + \log p$ **6.** $x - yp = ap^2$

7. $x^2\left(\frac{dy}{dx}\right)^4 + 2x\frac{dy}{dx} - y = 0.$ **8.** $y = 2px - xp^2$

Answers

1. $2y = cx^2 + \frac{a}{c}$ **2.** $y = 2\sqrt{cx} + \tan^{-1} c$ **3.** $16 + 2c^2y - c^3x^2 = 0$

4. $x = \log\frac{p-1}{\sqrt{p^2+1}} - \tan^{-1} p + c,\ y = \log\frac{p-1}{\sqrt{p^2+1}} + \tan^{-1} p + c$ **5.** $y = 3x + \log\frac{3}{1 - ce^{3x}}$

6. $x = \frac{p}{\sqrt{1-p^2}}(c + a\sin^{-1} p),\ y = \frac{1}{\sqrt{1-p^2}}(c + a\sin^{-1} p) - ap.$ **7.** $y = c^2 + 2\sqrt{cx}$.

8. $y = 2\sqrt{cx} - c.$

EQUATIONS SOLVABLE FOR *x*

If the equation is solvable for x, we can express x explicitly in terms of y and p. Thus, the equations of this type can be put as $x = f(y, p)$
...(1)

Differentiating (1) w.r.t. y, we get $$\frac{dx}{dy} = \frac{1}{p} = F\left(y, p, \frac{dp}{dy}\right) \qquad ...(2)$$

Equation (2) is a differential equation of first order in p and y.

Suppose the solution of (2) is $\phi(y, p, c) = 0$ ...(3)

Now elimination of p from (1) and (3) gives the required solution.

If p cannot be easily eliminated, then we solve equations (1) and (3) for x and y to get

$$x = \phi_1(p, c),\ y = \phi_2(p, c)$$

These two relations together constitute the solution of the given equation with p as parameter.

SOLVED EXAMPLES

Example 20. *Solve* $y = 2px + y^2p^3$.

Sol. Solving for x, we have $x = \frac{1}{2}\left(\frac{y}{p} - y^2p^2\right)$

Differentiating both sides w.r.t. y

$$\frac{dx}{dy} = \frac{1}{p} = \frac{1}{2}\left(\frac{1}{p} - \frac{y}{p^2}\cdot\frac{dp}{dy} - 2yp^2 - 2y^2p\frac{dp}{dy}\right)$$

or $$2p = p - y\frac{dp}{dy} - 2yp^4 - 2y^2p^3\frac{dp}{dy}$$

or $$p + 2yp^4 + y\frac{dp}{dy} + 2y^2p^3\frac{dp}{dy} = 0 \text{ or } p(1 + 2yp^3) + y\frac{dp}{dy}(1 + 2yp^3) = 0$$

or $$\left(p + y\frac{dp}{dy}\right)(1 + 2yp^3) = 0$$

Discarding the factor $(1 + 2yp^3)$ which does not involve $\frac{dp}{dy}$, we have

$$p + y\frac{dp}{dy} = 0 \quad \text{or} \quad \frac{dy}{y} + \frac{dp}{p} = 0$$

Integrating, $\log y + \log p = \log c$ or $py = c$ or $p = \frac{c}{y}$

Putting this value of p in the given equation, we have

$$y = \frac{2cx}{y} + \frac{c^3}{y} \quad \text{or} \quad y^2 = 2cx + c^3$$

which is the required solution.

Example 21. *Solve* $p = \tan\left(x - \frac{p}{1+p^2}\right)$.

Sol. Solving for x, we have $x = \tan^{-1} p + \frac{p}{1+p^2}$ ...(1)

Differentiating both sides w.r.t. y,

$$\frac{dx}{dy} = \frac{1}{p} = \frac{1}{1+p^2}\cdot\frac{dp}{dy} + \frac{(1+p^2) - 2p^2}{(1+p^2)^2}\cdot\frac{dp}{dy}$$

or $$\frac{1}{p} = \frac{2(1+p^2) - 2p^2}{(1+p^2)^2}\frac{dp}{dy} \quad \text{or} \quad dy = \frac{2p}{(1+p^2)^2}dp$$

Integrating, $y = c - \frac{1}{1+p^2}$ ...(2)

Equations (1) and (2) together constitute the general solution.

EXERCISE E

Solve the following equations:

1. $y = 3px + 6p^2y^2$
2. $y = 2px + p^2y$
3. $p^3 - 4xyp + 8y^2 = 0$
4. $y^2 \log y = xyp + p^2$
5. $x = y + a \log p$
6. $x = y + p^2$

Answers

1. $y^3 = 3cx + 6c^2$ 2. $y^2 = 2cy + c^2$ 3. $64y = c(c - 4x)^2$

4. $\log y = cx + c^2$ 5. $x = c + a \log \dfrac{p}{p-1}$, $y = c - a \log (p - 1)$

6. $x = -2p - 2 \log (1 - p) + c$, $y = -p^2 - 2p - 2 \log (1 - p) + c$.

CLAIRAUT'S EQUATION

An equation of the form $y = px + f(p)$ ...(1)
is known as Clairaut's equation.

Differentiating (1) w.r.t. x, we get

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx} \quad \text{or} \quad [x + f'(p)]\frac{dp}{dx} = 0$$

Discarding the factor $[x + f'(p)]$, we have $\dfrac{dp}{dx} = 0$

Integrating, $p = c$

Putting $p = c$ in (1), the required solution is $p = cx + f(c)$

Thus, the solution of Clairaut's equation is obtained by writing c for p.

SOLVED EXAMPLES

Example 22. *Solve* $(y - px)(p - 1) = p$.

Sol. The given equation can be written as

$$y - px = \frac{p}{p-1} \quad \text{or} \quad y = px + \frac{p}{p-1}$$

This is of Clairaut's form. Hence putting c for p, the solution is $y = cx + \dfrac{c}{c-1}$.

Note. Many differential equations can be reduced to Clairaut's form by suitably changing the variables.

Example 23. *Solve* $e^{4x}(p - 1) + e^{2y}p^2 = 0$.

Sol. [In problems involving e^{lx} and e^{my}, put $X = e^{kx}$ and $Y = e^{ky}$, where k is the H.C.F. of l and m].

Put $X = e^{2x}$ and $Y = e^{2y}$

so that $dX = 2e^{2x}\,dx$ and $dY = 2e^{2y}\,dy$

$$\therefore \quad p = \frac{dy}{dx} = \frac{e^{2x}}{e^{2y}}\frac{dY}{dX} = \frac{X}{Y}P, \quad \text{where } P = \frac{dY}{dX}$$

The given equation becomes $X^2\left(\frac{X}{Y}P - 1\right) + Y \cdot \frac{X^2}{Y^2}P^2 = 0.$

or $\qquad XP - Y + P^2 = 0 \quad \text{or} \quad Y = PX + P^2$ which is of Clairaut's form.

$\therefore$ Its solution is $Y = cX + c^2$ and hence $e^{2y} = ce^{2x} + c^2$.

Example 24. *Solve $(px - y)(py + x) = 2p$.*

Sol. Put $\quad X = x^2$ and $Y = y^2 \quad$ so that $\quad dX = 2x\,dx \quad$ and $\quad dY = 2y dy$

$$\therefore \qquad p = \frac{dy}{dx} = \frac{x}{y} \cdot \frac{dY}{dX} = \frac{\sqrt{X}}{\sqrt{Y}}P, \quad \text{where } P = \frac{dY}{dX}$$

The given equation becomes

$$\left(\frac{\sqrt{X}}{\sqrt{Y}}P \cdot \sqrt{X} - \sqrt{Y}\right)\left(\frac{\sqrt{X}}{\sqrt{Y}}P \cdot \sqrt{Y} + \sqrt{X}\right) = 2\frac{\sqrt{X}}{\sqrt{Y}}P$$

or $\qquad (PX - Y)(P + 1) = 2P \quad \text{or} \quad PX - Y = \dfrac{2P}{P+1}$

or $\qquad Y = PX - \dfrac{2P}{P+1}$ which is of Clairaut's form.

$\therefore$ Its solution is $Y = cX - \dfrac{2c}{c+1}$ and hence $y^2 = cx^2 - \dfrac{2c}{c+1}$.

EXERCISE F

Solve the following equations:

1. $y = xp + \dfrac{a}{p}$
2. $y = px + \sqrt{a^2p^2 + b^2}$
3. $\sin px \cos y = \cos px \sin y + p$
4. $xp^2 - yp + a = 0$
5. $(x - a)p^2 + (x - y)p - y = 0$
6. $p = \log(px - y)$
7. $p = \sin(y - px)$
8. $p^2(x^2 - 1) - 2pxy + y^2 - 1 = 0$
9. $e^{3x}(p - 1) + p^3e^{2y} = 0$
10. $x^2(y - px) = yp^2$
11. $(y + px)^2 = x^2p$.

Answers

1. $y = cx + \dfrac{a}{c}$
2. $y = cx + \sqrt{a^2c^2 + b^2}$
3. $y = cx - \sin^{-1} c$
4. $y = cx + \dfrac{a}{c}$
5. $y = cx - \dfrac{ac^2}{c+1}$
6. $y = cx - e^c$
7. $y = cx + \sin^{-1} c$
8. $(y - cx)^2 = 1 + c^2$
9. $e^y = ce^x + c^2$
10. $y^2 = cx^2 + c^2$ [**Hint.** Put $x^2 = X$, $y^2 = Y$]
11. $xy = cx - c^2$. [**Hint.** Put $xy = v$]

4. LINEAR DIFFERENTIAL EQUATIONS OF SECOND AND HIGHER ORDER

STRUCTURE

DEFINITIONS

A **linear differential equation** is that in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together. Thus, the general linear differential equation of the n^{th} order is of the form

$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots\cdots + P_{n-1}\frac{dy}{dx} + P_n y = X$, where $P_1, P_2, \ldots\ldots, P_{n-1}, P_n$ and X are functions of x only.

A **linear differential equation with constant co-efficients** is of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots\cdots + a_{n-1} \frac{dy}{dx} + a_n y = X \quad \ldots(1)$$

where $a_1, a_2, \ldots\ldots, a_{n-1}, a_n$ are constants and X is either a constant or a function of x only.

THE OPERATOR D

The part $\frac{d}{dx}$ of the symbol $\frac{dy}{dx}$ may be regarded as an operator such that when it operates on y, the result is the derivative of y.

Similarly, $\frac{d^2}{dx^2}, \frac{d^3}{dx^3}, \ldots\ldots, \frac{d^n}{dx^n}$ may be regarded as operators.

For brevity, we write $\frac{d}{dx} = D, \frac{d^2}{dx^2} = D^2, \ldots\ldots, \frac{d^n}{dx^n} = D^n$

Thus, the symbol D is a **differential operator** or simply an **operator.**

Written in symbolic form, equation (1) becomes

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots\cdots + a_{n-1} D + a_n) y = X$$

or $$f(D) y = X$$

where $f(D) = D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots\cdots + a_{n-1} D + a_n$

i.e., $f(D)$ is a polynomial in D.

The operator D can be treated as an algebraic quantity.

Thus $$D(u + v) = Du + Dv$$

$$D(\lambda u) = \lambda\, Du$$

$$D^p D^q\, u = D^{p+q}\, u$$

$$D^p D^q u = D^q D^p u$$

The polynomial $f(D)$ can be factorised by ordinary rules of algebra and the factors may be written in any order.

THEOREMS

Theorem 1

If $y = y_1, y = y_2, \ldots, y = y_n$ are n linearly independent solutions of the differential equation

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n) y = 0 \quad \ldots(i)$$

then $u = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ is also its solution, where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

NOTES

Proof. Since $y = y_1, y = y_2, ..., y = y_n$ are solution of equation (*i*).

$$\therefore \quad \left.\begin{aligned} &D^n y_1 + a_1 D^{n-1} y_1 + a_2 D^{n-2} y_1 + \cdots + a_n y_1 = 0 \\ &D^n y_2 + a_1 D^{n-1} y_2 + a_2 D^{n-2} y_2 + \cdots + a_n y_2 = 0 \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ &D^n y_n + a_1 D^{n-1} y_n + a_2 D^{n-2} y_n + \cdots + a_n y_n = 0 \end{aligned}\right\} \quad \ldots(ii)$$

Now

$$\begin{aligned} &D^n u + a_1 D^{n-1} u + a_2 D^{n-2} u + \cdots + a_n u \\ &= D^n (c_1 y_1 + c_2 y_2 + \cdots + c_n y_n) \\ &\quad + a_1 D^{n-1} (c_1 y_1 + c_2 y_2 + \cdots + c_n y_n) \\ &\quad + a_2 D^{n-2} (c_1 y_1 + c_2 y_2 + \cdots + c_n y_n) \\ &\quad + \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ &\quad + a_n (c_1 y_1 + c_2 y_2 + \cdots + c_n y_n) \\ &= c_1 (D^n y_1 + a_1 D^{n-1} y_1 + a_2 D^{n-2} y_1 + \cdots + a_n y_1) \\ &\quad + c_2 (D^n y_2 + a_1 D^{n-1} y_2 + a_2 D^{n-2} y_n + \cdots + a_n y_2) \\ &\quad + \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ &\quad + c_n (D^n y_n + a_1 D^{n-1} y_n + a_2 D^{n-2} y_2 + \cdots + a_n y_n) \\ &= c_1(0) + c_2(0) + \cdots + c_n(0) \qquad [\because \text{ of } (ii)] \\ &= 0 \end{aligned}$$

which shows that $u = c_1 y_1 + c_2 y_2 + \cdots\cdots + c_n y_n$ is also the solution of equation (*i*).

Since this solution contains n arbitrary constants, it is the general or complete solution of equation (*i*).

Theorem 2

If $y = u$ is the complete solution of the equation $f(D)y = 0$ and $y = v$ is a particular solution (containing no arbitrary constants) of the equation $f(D)y = X$, then the complete solution of the equation $f(D)y = X$ is $y = u + v$.

Proof. Since $y = u$ is the complete solution of the equation $f(D)y = 0$ ...(*i*)

$\therefore$ $f(D)u = 0$ ...(*ii*)

Also $y = v$ is a particular solution of the equation $f(D)y = X$ ...(*iii*)

$\therefore$ $f(D)v = X$ ...(*iv*)

Adding (*ii*) and (*iv*), we have $f(D)(u + v) = X$

Thus $y = u + v$ satisfies the equation (*iii*), hence it is the **complete solution (C.S.)** because it contains n arbitrary constants.

The part $y = u$ is called the **complementary function (C.F.)** and the part $y = v$ is called the **particular integral (P.I.)** of the equation (*iii*).

$\therefore$ The complete solution of equation (*iii*), is **y = C.F. + P.I.**

Thus in order to solve the equation (iii), we first find the C.F. i.e., the C.S. of equation (i) and then the P.I. i.e., a particular solution of equation (iii).

AUXILIARY EQUATION (A.E.)

Consider the differential equation

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots\cdots + a_n) y = 0 \qquad \ldots(i)$$

Let $y = e^{mx}$ be a solution of (*i*), then $Dy = me^{mx}$, $D^2y = m^2e^{mx}$,......, $D^{n-2}y = m^{n-2}e^{mx}$

$$D^{n-1}y = m^{n-1}e^{mx},\ D^ny = m^ne^{mx}$$

Substituting the values of y, Dy, D^2y, ..., D^ny in (*i*), we get

$$(m^n + a_1m^{n-1} + a_2m^{n-2} + \cdots\cdots + a_n)\, e^{mx} = 0$$

or $$m^n + a_1m^{n-1} + a_2m^{n-2} + \cdots\cdots + a_n = 0, \text{ since } e^{mx} \neq 0 \qquad \ldots(ii)$$

Thus $y = e^{mx}$ will be a solution of equation (*i*) if m satisfies equation (*ii*).

Equation (*ii*) is called the auxiliary equation for the differential equation (*i*).

Replacing m by D in (*ii*), we get $D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots\cdots + a_n = 0$ ...(*iii*)

Equation (*ii*) gives the same values of m *as* equation (*iii*) gives of D. In practice, we take equation (*iii*) as the auxiliary equation which is obtained by equating to zero the symbolic co-efficient of y in equation (*i*).

Definition. The equation obtained by equating to zero the symbolic co-efficient of y is called the **auxiliary equation**, briefly written as **A.E.**

RULES FOR FINDING THE COMPLEMENTARY FUNCTION

Consider the equation $(D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots\cdots + a_n)y = 0$ ...(*i*)

where all the a_i's are constant.

Its auxiliary equation is $D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots\cdots + a_n = 0$ ...(*ii*)

Let $D = m_1, m_2, m_3, \ldots\ldots, m_n$ be the roots of the A.E. The solution of equation (*i*) depends upon the nature of roots of the A.E. The following cases arise:

Case I. *If all the roots of the A.E. are real and distinct*, then equation (*ii*) is equivalent to

$$(D - m_1)(D - m_2)\ \ldots\ldots\ (D - m_n) = 0 \qquad \ldots(iii)$$

Equation (*iii*) will be satisfied by the solutions of the equations

$$(D - m_1)y = 0,\ (D - m_2)y = 0, \ldots\ldots, (D - m_n)y = 0$$

Now, consider the equation $(D - m_1)y = 0$, *i.e.*, $\dfrac{dy}{dx} - m_1y = 0$

It is a linear equation and I.F. $= e^{\int -m_1 dx} = e^{-m_1x}$

$\therefore$ its solution is $y \,.\, e^{-m_1x} = \int 0 \,.\, e^{-m_1x}\, dx + c_1$ or $y = c_1e^{m_1x}$

Similarly, the solution of $(D - m_2)y = 0$ is $y = c_2e^{m_2x}$

....

the solution of $(D - m_n)y = 0$ is $y = c_ne^{m_nx}$

Hence the complete solution of equation (*i*) is

$$y = c_1e^{m_1x} + c_2e^{m_2x} + \cdots\cdots + c_ne^{m_nx} \qquad \ldots(iv)$$

Case II. *If two roots of the A.E. are equal*, let $m_1 = m_2$.

The solution obtained in equation (*iv*) becomes

$$y = (c_1 + c_2)\, e^{m_1x} + c_3e^{m_3x} + \cdots\cdots + c_ne^{m_nx}$$

$$= ce^{m_1x} + c_3e^{m_3x} + \cdots\cdots + c_ne^{m_nx}$$

It contains $(n-1)$ arbitrary constants and is, therefore, not the complete solution of equation (*i*).

The part of the complete solution corresponding to the repeated root is the complete solution of

$$(D - m_1)(D - m_1)y = 0$$

Putting $(D - m_1)y = v$, it becomes $(D - m_1)v = 0$ *i.e.*, $\frac{dv}{dx} - m_1 v = 0$

As in case I, its solution is $v = c_1 e^{m_1 x}$

$$\therefore \qquad (D - m_1)y = c_1 e^{m_1 x} \quad \text{or} \quad \frac{dy}{dx} - m_1 y = c_1 e^{m_1 x}$$

which is a linear equation and I.F. = $e^{-m_1 x}$

$\therefore$ its solution is $y \cdot e^{-m_1 x} = \int c_1 e^{m_1 x} \cdot e^{-m_1 x}\, dx + c_2 = c_1 x + c_2$

or
$$y = (c_1 x + c_2)\, e^{m_1 x}$$

Thus, the complete solution of equation (*i*) is

$$y = (c_1 x + c_2)\, e^{m_1 x} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

If, however, three roots of the A.E. are equal, say $m_1 = m_2 = m_3$, then proceeding as above, the solution becomes

$$y = (c_1 x^2 + c_2 x + c_3)\, e^{m_1 x} + c_4 e^{m_4 x} + \ldots\ldots + c_n e^{m_n x}$$

Case III. *If two roots of the A.E. are imaginary*, let

$$m_1 = \alpha + i\beta \quad \text{and} \quad m_2 = \alpha - i\beta$$

The solution obtained in equation (*iv*) becomes

$$y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

$$= e^{\alpha x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x}) + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

$$= e^{\alpha x}[c_1(\cos \beta x + i \sin \beta x) + c_2(\cos \beta x - i \sin \beta x)]$$

$$+ c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

[$\because$ By Euler's Theorem, $e^{i\theta} = \cos\theta + i \sin\theta$]

$$= e^{\alpha x}[(c_1 + c_2)\cos \beta x + i(c_1 - c_2)\sin \beta x] + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

$$= e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x) + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

[Taking $c_1 + c_2 = C_1$, $i(c_1 - c_2) = C_2$]

Case IV. *If two pairs of imaginary roots be equal*, let

$$m_1 = m_2 = \alpha + i\beta \quad \text{and} \quad m_3 = m_4 = \alpha - i\beta$$

Then by case II, the complete solution is

$$y = e^{\alpha x}[(c_1 x + c_2)\cos \beta x + (c_3 x + c_4)\sin \beta x] + c_5 e^{m_5 x} + \ldots\ldots + c_n e^{m_n x}.$$

SOLVED EXAMPLES

Example 1. *Solve:* $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} - 6y = 0$.

Sol. Given equation in symbolic form is $(D^3 - 7D - 6)y = 0$

Its A.E. is $D^3 - 7D - 6 = 0$ or $(D + 1)(D + 2)(D - 3) = 0$

whence $D = -1, -2, 3$

Hence the C.S. is $y = c_1e^{-x} + c_2e^{-2x} + c_3e^{3x}$.

Example 2. *Solve:* $(D^3 - 4D^2 + 4D)y = 0$.

Sol. The A.E. is $D^3 - 4D^2 + 4D = 0$ or $D(D^2 - 4D + 4) = 0$

or $D(D - 2)^2 = 0$

whence $D = 0, 2, 2$

Hence, the C.S is $y = c_1e^{0x} + (c_2x + c_3)e^{2x}$ or $y = c_1 + (c_2x + c_3)e^{2x}$.

Example 3. *Solve:* $\frac{d^4y}{dx^4} + 13\frac{d^2y}{dx^2} + 36y = 0$.

Sol. Given equation in symbolic form is $(D^4 + 13D^2 + 36)y = 0$

Its A.E. is $D^4 + 13D^2 + 36 = 0$

or $(D^2 + 4)(D^2 + 9) = 0$ $\therefore$ $D = \pm 2i, \pm 3i$

Hence the C.S. is $y = e^{0x}(c_1 \cos 2x + c_2 \sin 2x) + e^{0x}(c_3 \cos 3x + c_4 \sin 3x)$

or $y = c_1 \cos 2x + c_2 \sin 2x + c_3 \cos 3x + c_4 \sin 3x$.

Example 4. *Solve:* $\frac{d^4x}{dt^4} + 4x = 0$.

Sol. Given equation in symbolic form is $(D^4 + 4)x = 0$, where $D = \frac{d}{dt}$

Its A.E. is $D^4 + 4 = 0$ or $(D^4 + 4D^2 + 4) - 4D^2 = 0$

or $(D^2 + 2)^2 - (2D)^2 = 0$ or $(D^2 + 2D + 2)(D^2 - 2D + 2) = 0$

whence $D = \frac{-2 \pm \sqrt{-4}}{2}$ and $\frac{2 \pm \sqrt{-4}}{2}$ *i.e.,* $D = -1 \pm i$ and $1 \pm i$

Hence the C.S. is $x = e^{-t}(c_1 \cos t + c_2 \sin t) + e^t(c_3 \cos t + c_4 \sin t)$.

Example 5. *Solve:* $y'' - 2y' + 10y = 0$, *given* $y(0) = 4$, $y'(0) = 1$.

Sol. Given equation in symbolic form is

$$(D^2 - 2D + 10)y = 0$$

Its A.E. is $D^2 - 2D + 10 = 0$

$\Rightarrow$ $D = \frac{2 \pm \sqrt{4 - 40}}{2} = \frac{2 \pm 6i}{2} = 1 \pm 3i$

The C.S. is $y = e^x(c_1 \cos 3x + c_2 \sin 3x)$ ...(1)

Now $y(0) = 4$ $\Rightarrow$ $y = 4$, when $x = 0$

$\therefore$ $4 = c_1$

Equation (1) becomes $y = e^x(4 \cos 3x + c_2 \sin 3x)$ ...(2)

so that $y' = e^x(4 \cos 3x + c_2 \sin 3x) + e^x(-12 \sin 3x + 3c_2 \cos 3x)$

Since $y'(0) = 1$ *i.e.,* $y' = 1$, when $x = 0$

$\therefore$ $1 = 4 + 3c_2$ $\Rightarrow$ $c_2 = -1$

Equation (2) becomes $y = e^x(4 \cos 3x - \sin 3x)$, which is the required particular solution.

NOTES

EXERCISE A

Solve the following differential equations:

1. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} - 5y = 0$.
2. $\dfrac{d^2y}{dx^2} + (a+b)\dfrac{dy}{dx} + aby = 0$.
3. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + y = 0$.
4. $\dfrac{d^2x}{dt^2} + 8\dfrac{dx}{dt} + 16x = 0$.
5. $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - y = 0$.
6. $\dfrac{d^3y}{dx^3} + 6\dfrac{d^2y}{dx^2} + 11\dfrac{dy}{dx} + 6y = 0$.
7. $\dfrac{d^4y}{dx^4} - 5\dfrac{d^2y}{dx^2} + 4y = 0$.
8. $\dfrac{d^4y}{dx^4} + 6\dfrac{d^2y}{dx^2} + 9y = 0$.
9. $(D^2 + 1)^3 (D^2 + D + 1)^2 y = 0$.
10. $\dfrac{d^3y}{dx^3} + y = 0$.
11. $\dfrac{d^2y}{dx^2} + y = 0$, given that $y(0) = 2$ and $y\left(\dfrac{\pi}{2}\right) = -2$.
12. $\dfrac{d^2x}{dt^2} - 3\dfrac{dx}{dt} + 2x = 0$, given that, when $t = 0$, $x = 0$ and $\dfrac{dx}{dt} = 0$.
13. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 29y = 0$, given that, when $x = 0$, $y = 0$ and $\dfrac{dy}{dx} = 15$.
14. If $\dfrac{d^4x}{dt^4} = m^4x$, show that $x = c_1 \cos mt + c_2 \sin mt + c_3 \cosh mt + c_4 \sinh mt$.
15. Solve the differential equation: $9y''' + 3y'' - 5y' + y = 0$.
16. Solve the differential equation $\dfrac{d^3y}{dx^3} + 6\dfrac{d^2y}{dx^2} + 12\dfrac{dy}{dx} + 8y = 0$ under the conditions $y(0) = 0$, $y'(0) = 0$ and $y''(0) = 2$.
17. Solve the differential equation $\dfrac{d^2i}{dt^2} + \dfrac{R}{L}\dfrac{di}{dt} + \dfrac{i}{LC} = 0$, where $R^2C = 4L$ and R, C, L are constants.

Answers

1. $y = c_1 e^{5x} + c_2e^{-x}$
2. $y = c_1 e^{-ax} + c_2e^{-bx}$
3. $y = c_1e^{(2+\sqrt{3})x} + c_2e^{(2-\sqrt{3})x}$
4. $x = (c_1 + c_2t)\, e^{-4t}$
5. $y = (c_1 + c_2x + c_3x^2)\, e^x$
6. $y = c_1e^{-x} + c_2e^{-2x} + c_3e^{-3x}$
7. $y = c_1e^x + c_2e^{-x} + c_3e^{2x} + c_4e^{-2x}$
8. $y = (c_1 + c_2x) \cos \sqrt{3}\, x + (c_3 + c_4x) \sin \sqrt{3}\, x$
9. $y = (c_1 + c_2x + c_3x^2) \cos x + (c_4 + c_5x + c_6x^2) \sin x$
$$+ e^{-\frac{1}{2}x}\left[(c_7 + c_8x)\cos\frac{\sqrt{3}}{2}x + (c_9 + c_{10}x)\sin\frac{\sqrt{3}}{2}x\right]$$
10. $y = c_1e^{-x} + e^{x/2}\left(c_2\cos\dfrac{\sqrt{3}x}{2} + c_3\sin\dfrac{\sqrt{3}x}{2}\right)$
11. $y = 2(\cos x - \sin x)$
12. $x = 0$
13. $y = 3e^{-2x}\sin 5x$
15. $y = c_1e^{-x} + (c_2 + c_3x)\, e^{\frac{1}{3}x}$
16. $y = x^2e^{-2x}$
17. $i = (c_1 + c_2t)\, e^{-\frac{Rt}{2L}}$.

NOTES

THE INVERSE OPERATOR $\dfrac{1}{f(D)}$

Definition. $\dfrac{1}{f(D)}$ X is that function of x, *free from arbitrary constants,* which when operated upon by f(D) gives X.

Thus $\quad f(D)\left\{\dfrac{1}{f(D)}X\right\} = X$

$\therefore$ f(D) and $\dfrac{1}{f(D)}$ are inverse operators.

Theorem 1. $\dfrac{1}{f(D)}$ *X is the particular integral of f(D)y = X.*

Proof. The given equation is $f(D)y = X$ ...(1)

Putting $y = \dfrac{1}{f(D)}X$ in (1), we have

$$f(D)\left\{\frac{1}{f(D)}X\right\} = X \quad \text{or} \quad X = X$$

which is true.

$$\therefore \qquad y = \frac{1}{f(D)}X \text{ is a solution of (1).}$$

Since it contains no arbitrary constants, it is the particular integral of $f(D)\,y = X$.

Theorem 2. $\dfrac{1}{D}X = \int X\,dx.$

Proof. Let $\dfrac{1}{D}X = y$

Operating both sides by D, we have $D\left(\dfrac{1}{D}X\right) = Dy$ or $X = \dfrac{dy}{dx}$

Integrating both sides w.r.t. x

$$y = \int X\,dx,$$

no arbitrary constant being added since $y = \dfrac{1}{D}X$ contains no arbitrary constant.

$$\therefore \qquad \frac{1}{D}X = \int X\,dx.$$

Theorem 3. $\dfrac{1}{D-a}X = e^{ax}\int Xe^{-ax}\,dx.$

Proof. Let $\dfrac{1}{D-a}X = y$

Operating on both sides by (D – a), $(D-a)\left(\frac{1}{D-a}X\right) = (D-a)y$

or $$X = \frac{dy}{dx} - ay \quad i.e., \quad \frac{dy}{dx} - ay = X$$

which is a linear equation and I.F. $= e^{\int -a\,dx} = e^{-ax}$

∴ Its solution is $ye^{-ax} = \int X e^{-ax}\, dx$, no constant being added

or $$y = e^{ax} \int X e^{-ax}\, dx$$

Hence, $$\frac{1}{D-a} X = e^{ax} \int e^{-ax} X\, dx.$$

RULES FOR FINDING THE PARTICULAR INTEGRAL

Consider the differential equation,

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + ... + a_{n-1}D + a_n)y = X$$

It can be written as $f(D)y = X$

∴ $$\text{P.I.} = \frac{1}{f(D)} X$$

Case I. *When* $X = e^{ax}$

Since,
$$D\, e^{ax} = a\, e^{ax}$$
$$D^2 e^{ax} = a^2 e^{ax}$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$D^{n-1} e^{ax} = a^{n-1} e^{ax}$$
$$D^n e^{ax} = a^n e^{ax}$$

∴ $$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots\cdots + a_{n-1}D + a_n)\, e^{ax}$$
$$= (a^n + a_1 a^{n-1} + a_2 a^{n-2} + \cdots\cdots + a_{n-1} a + a_n)\, e^{ax}$$

or $$f(D)\, e^{ax} = f(a)\, e^{ax}$$

Operating on both sides by $\frac{1}{f(D)}$.

$$\frac{1}{f(D)}(f(D)e^{ax}) = \frac{1}{f(D)}(f(a)e^{ax}) \quad \text{or} \quad e^{ax} = f(a)\frac{1}{f(D)} e^{ax}$$

Dividing both sides by $f(a)$, $\frac{1}{f(a)} e^{ax} = \frac{1}{f(D)} e^{ax}$, provided $f(a) \neq 0$

Hence, $$\boldsymbol{\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}}, \textbf{ provided } \boldsymbol{f(a) \neq 0.}$$

Case of failure. If $f(a) = 0$, the above method fails.

Since $f(a) = 0$, D = a is a root of A.E. $f(D) = 0$

∴ D – a is a factor of f(D).

Let $$f(D) = (D-a)\,\phi(D), \text{ where } \phi(a) \neq 0 \qquad ...(i)$$

Then $$\frac{1}{f(\mathrm{D})}e^{ax} = \frac{1}{(\mathrm{D}-a)\,\phi(\mathrm{D})}e^{ax} = \frac{1}{\mathrm{D}-a}\cdot\frac{1}{\phi(\mathrm{D})}e^{ax} = \frac{1}{\mathrm{D}-a}\cdot\frac{1}{\phi(a)}e^{ax}$$

$$= \frac{1}{\phi(a)}\cdot\frac{1}{\mathrm{D}-a}e^{ax} = \frac{1}{\phi(a)}e^{ax}\int e^{ax}\,.\,e^{-ax}dx$$ [By Theorem 3]

$$= \frac{1}{\phi(a)}e^{ax}\int 1\,dx = x\,.\,\frac{1}{\phi(a)}e^{ax} \qquad ...(ii)$$

Differentiating both sides of (*i*) w.r.t. D, we have $f'(\mathrm{D}) = (\mathrm{D}-a)\,\phi'(\mathrm{D}) + \phi(\mathrm{D})$

$\Rightarrow \qquad f'(a) = \phi(a)$

$\therefore$ From (*ii*), we have $\boldsymbol{\dfrac{1}{f(\mathrm{D})}e^{ax} = x\,.\,\dfrac{1}{f'(a)}e^{ax}}$, **provided $f'(a) \neq 0$**

If $f'(a) = 0$, then $\dfrac{1}{f(\mathrm{D})}e^{ax} = x^2\,.\,\dfrac{1}{f''(a)}e^{ax}$, provided $f''(a) \neq 0$

and so on.

SOLVED EXAMPLES

Example 6. *Find the P.I. of $(4D^2 + 4D - 3)\,y = e^{2x}$.*

Sol. $$\text{P.I.} = \frac{1}{4\mathrm{D}^2 + 4\mathrm{D} - 3}e^{2x} = \frac{1}{4(2)^2 + 4(2) - 3}e^{2x} \quad \text{(replacing D by 2)}$$

$$= \frac{1}{21}e^{2x}.$$

Example 7. *Find the P.I. of $(D^2 + 3D + 2)y = 5$.*

Sol. $$\text{P.I.} = \frac{1}{\mathrm{D}^2 + 3\mathrm{D} + 2}(5e^{0x}) \qquad [\because \; e^{0x} = 1]$$

$$= 5\,.\,\frac{1}{0+0+2}e^{0x} \qquad \text{(replacing D by 0)}$$

$$= \frac{5}{2}.$$

Example 8. *Find the P.I. of $(D^3 - 3D^2 + 4)y = e^{2x}$.*

Sol. $$\text{P.I.} = \frac{1}{\mathrm{D}^3 - 3\mathrm{D}^2 + 4}e^{2x}.$$

Here the denom. vanishes, when D is replaced by 2. It is a case of failure.

We multiply the numerator by x and differentiate the denominator w.r.t. D.

$$\therefore \qquad \text{P.I.} = x\,.\,\frac{1}{3\mathrm{D}^2 - 6\mathrm{D}}e^{2x}$$

It is again a case of failure. We multiply the numerator by x and differentiate the denominator w.r.t. D.

$$\therefore \qquad \text{P.I.} = x^2\,.\,\frac{1}{6\mathrm{D} - 6}e^{2x} = x^2\,.\,\frac{1}{6(2) - 6}e^{2x} = \frac{x^2}{6}e^{2x}.$$

NOTES

Case II. *When X = sin (ax + b) or cos (ax + b)*

$$D \sin(ax+b) = a\cos(ax+b)$$

$$\mathbf{D^2 \sin(ax+b) = (-a^2)\sin(ax+b)}$$

$$D^3 \sin(ax+b) = -a^3\cos(ax+b)$$

$$D^4 \sin(ax+b) = a^4 \sin(ax+b)$$

or $$\mathbf{(D^2)^2 \sin(ax+b) = (-a^2)^2 \sin(ax+b)}$$

....

....

In general, $(D^2)^n \sin(ax+b) = (-a^2)^n \sin(ax+b)$

$\therefore$ $f(D^2)\sin(ax+b) = f(-a^2)\sin(ax+b)$

Operating on both sides by $\frac{1}{f(D^2)}$,

$$\frac{1}{f(D^2)}(f(D^2)\sin(ax+b)) = \frac{1}{f(D^2)}[f(-a^2)\sin(ax+b)]$$

or $$\sin(ax+b) = f(-a^2)\frac{1}{f(D^2)}\sin(ax+b).$$

Dividing both sides by $f(-a^2)$,

$$\frac{1}{f(-a^2)}\sin(ax+b) = \frac{1}{f(D^2)}\sin(ax+b), \qquad \text{provided } f(-a^2) \neq 0.$$

Hence, $$\mathbf{\frac{1}{f(D^2)}\sin(ax+b) = \frac{1}{f(-a^2)}\sin(ax+b), \quad \text{provided } f(-a^2) \neq 0}$$

Similarly, $$\mathbf{\frac{1}{f(D^2)}\cos(ax+b) = \frac{1}{f(-a^2)}\cos(ax+b), \quad \text{provided } f(-a^2) \neq 0}$$

Case of Failure. If $f(-a^2) = 0$, the above method fails.

Since $\cos(ax+b) + i\sin(ax+b) = e^{i(ax+b)}$ | Euler's Theorem

$$\therefore \quad \frac{1}{f(D^2)}[\cos(ax+b) + i\sin(ax+b)] = \frac{1}{f(D^2)}e^{i(ax+b)}$$

[If we replace D by *ia*, $f(D^2) = f(-a^2) = 0$, so that it is a case of failure]

$$= x \cdot \frac{1}{f'(D^2)}e^{i(ax+b)} = x \cdot \frac{1}{f'(D^2)}[\cos(ax+b) + i\sin(ax+b)]$$

Equating real parts

$$\mathbf{\frac{1}{f(D^2)}\cos(ax+b) = x \cdot \frac{1}{f'(D^2)}\cos(ax+b), \qquad \text{provided } f'(-a^2) \neq 0}$$

Equating imaginary parts

$$\mathbf{\frac{1}{f(D^2)}\sin(ax+b) = x \cdot \frac{1}{f'(D^2)}\sin(ax+b), \qquad \text{provided } f'(-a^2) \neq 0}$$

If $f'(-a^2) = 0$, then

$$\frac{1}{f(D^2)}\sin(ax+b) = x^2 \cdot \frac{1}{f''(D^2)}\sin(ax+b), \qquad \text{provided } f''(-a^2) \neq 0$$

$$\frac{1}{f(D^2)}\cos(ax+b) = x^2 \cdot \frac{1}{f''(D^2)}\cos(ax+b), \qquad \text{provided } f''(-a^2) \neq 0$$

and so on.

Example 9. *Find the P.I. of $(D^3 + 1)y = \sin(2x + 3)$.*

Sol. $$\text{P.I.} = \frac{1}{D^3+1}\sin(2x+3) = \frac{1}{D(-2^2)+1}\sin(2x+3)$$ [Putting $D^2 = -2^2$]

$$= \frac{1}{1-4D}\sin(2x+3)$$

Multiplying and dividing by (1 + 4D)

$$= \frac{1+4D}{(1-4D)(1+4D)}\sin(2x+3) = \frac{1+4D}{1-16D^2}\sin(2x+3)$$

$$= \frac{1+4D}{1-16(-2^2)}\sin(2x+3)$$ [Putting $D^2 = -2^2$]

$$= \frac{1}{65}[\sin(2x+3) + 4D\sin(2x+3)]$$

$$= \frac{1}{65}[\sin(2x+3) + 8\cos(2x+3)] \qquad \left[\because\ D = \frac{d}{dx}\right]$$

Example 10. *Find the P.I. of $(D^2 + 4)y = \cos 2x$.*

Sol. $$\text{P.I.} = \frac{1}{D^2+4}\cos 2x$$

Here the denominator vanishes when D is replaced by $-2^2 = -4$. It is a case of failure. We multiply the numerator by x and differentiate the denominator w.r.t. D.

$$\therefore \quad \text{P.I.} = x \cdot \frac{1}{2D}\cos 2x = \frac{x}{2}\int \cos 2x\, dx \qquad \left[\because\ \frac{1}{D}f(x) = \int f(x)dx\right]$$

$$= \frac{x}{4}\sin 2x.$$

Case III. *When $X = x^m$, m* being a positive integer.

Here, $$\text{P.I.} = \frac{1}{f(D)}x^m$$

Take out the lowest degree term from $f(D)$ to make the first term unity (so that Binomial Theorem for a negative index is applicable).

The remaining factor will be of the form $1 + \phi(D)$ or $1 - \phi(D)$

Take this factor in the numerator. It takes the form

$$[1 + \phi(D)]^{-1} \quad \text{or} \quad [1 - \phi(D)]^{-1}$$

Expand it in ascending powers of D as far as the term containing D^m, since $D^{m+1}(x^m) = 0$, $D^{m+2}(x^m) = 0$ and so on.

Operate on x^m term by term.

Example 11. *Find the P.I. of $(D^2 + 5D + 4)y = x^2 + 7x + 9$.*

Sol. $$\text{P.I.} = \frac{1}{D^2+5D+4}(x^2+7x+9) = \frac{1}{4\left(1+\frac{5D}{4}+\frac{D^2}{4}\right)}(x^2+7x+9)$$

$$= \frac{1}{4}\left[1+\left(\frac{5D}{4}+\frac{D^2}{4}\right)\right]^{-1}(x^2+7x+9)$$

NOTES

$$= \frac{1}{4}\left[1 - \left(\frac{5D}{4} + \frac{D^2}{4}\right) + \left(\frac{5D}{4} + \frac{D^2}{4}\right)^2 - \ldots\ldots\right](x^2 + 7x + 9)$$

$$= \frac{1}{4}\left(1 - \frac{5D}{4} - \frac{D^2}{4} + \frac{25D^2}{16} \ldots\ldots\right)(x^2 + 7x + 9)$$

$$= \frac{1}{4}\left(1 - \frac{5D}{4} + \frac{21D^2}{16} \ldots\ldots\right)(x^2 + 7x + 9)$$

$$= \frac{1}{4}\left[(x^2 + 7x + 9) - \frac{5}{4}D(x^2 + 7x + 9) + \frac{21}{16}D^2(x^2 + 7x + 9)\right]$$

$$= \frac{1}{4}\left[(x^2 + 7x + 9) - \frac{5}{4}(2x + 7) + \frac{21}{16}(2)\right] = \frac{1}{4}\left(x^2 + \frac{9}{2}x + \frac{23}{8}\right).$$

Case IV. *When $X = e^{ax}V$, where V is a function of x.*

Let u be a function of x, then by successive differentiation, we have

$$D(e^{ax} u) = e^{ax} Du + a e^{ax} u = e^{ax} (D + a)u$$

$$D^2 (e^{ax} u) = D [e^{ax} (D + a) u] = e^{ax} (D^2 + aD) u + ae^{ax} (D + a)u$$

$$= e^{ax} (D^2 + 2aD + a^2) u = e^{ax} (D + a)^2 u$$

Similarly, $D^3 (e^{ax} u) = e^{ax} (D + a)^3 u$

In general, $D^n (e^{ax} u) = e^{ax} (D + a)^n u$

$\therefore$ $f(D) (e^{ax} u) = e^{ax} f(D + a) u$

Operating on both sides by $\frac{1}{f(D)}$,

$$\frac{1}{f(D)}[f(D)(e^{ax} u)] = \frac{1}{f(D)}[e^{ax} f(D + a)u]$$

$\Rightarrow$
$$e^{ax} u = \frac{1}{f(D)}[e^{ax} f(D + a)u] \qquad \ldots(i)$$

Now let $f(D + a) u = V$, *i.e.*, $u = \frac{1}{f(D + a)} V$

$\therefore$ From (*i*), we have $e^{ax} \frac{1}{f(D + a)} V = \frac{1}{f(D)}(e^{ax} V)$

or
$$\boldsymbol{\frac{1}{f(D)}(e^{ax} V) = e^{ax} \frac{1}{f(D + a)} V.}$$

Thus e^{ax} which is on the right of $\frac{1}{f(D)}$ may be taken out to the left provided D is replaced by D + *a*.

Example 12. *Find the P.I. of $(D^2 - 4D + 3)y = e^x \cos 2x$.*

Sol.
$$\text{P.I.} = \frac{1}{D^2 - 4D + 3} e^x \cos 2x = e^x \frac{1}{(D + 1)^2 - 4(D + 1) + 3} \cos 2x$$

$$= e^x \frac{1}{D^2 - 2D} \cos 2x = e^x \frac{1}{-2^2 - 2D} \cos 2x \qquad [\text{Putting } D^2 = -2^2]$$

$$= -\frac{1}{2} e^x \frac{1}{2 + D} \cos 2x = -\frac{1}{2} e^x \frac{2 - D}{(2 + D)(2 - D)} \cos 2x$$

$$= -\frac{1}{2}e^x \frac{2-D}{4-D^2}\cos 2x = -\frac{1}{2}e^x \frac{2-D}{4-(-2^2)}\cos 2x$$

$$= -\frac{1}{16}e^x (2\cos 2x - D\cos 2x) = -\frac{1}{16}e^x (2\cos 2x + 2\sin 2x)$$

$$= -\frac{1}{8}e^x(\cos 2x + \sin 2x).$$

Case V. *When X is any other function of x.*

Resolve $f(D)$ into linear factors.

Let $f(D) = (D - m_1)(D - m_2) \cdots\cdots (D - m_n)$

Then $\text{P.I.} = \frac{1}{f(D)}X = \frac{1}{(D-m_1)(D-m_2)\cdots\cdots(D-m_n)}X$

$$= \left(\frac{A_1}{D-m_1} + \frac{A_2}{D-m_2} + \cdots\cdots + \frac{A_n}{D-m_2}\right)X \quad \text{(Partial Fractions)}$$

$$= A_1\frac{1}{D-m_1}X + A_2\frac{1}{D-m_2}X + \cdots\cdots + A_n\frac{1}{D-m_n}X$$

$$= A_1 e^{m_1 x}\int X e^{-m_1 x}\,dx + A_2 e^{m_2 x}\int X e^{-m_2 x}dx + \cdots\cdots + A_n e^{m_n x}\int X e^{-m_n x}dx$$

$$\left[\because \quad \frac{1}{D-m}X = e^{mx}\int Xe^{-mx}dx\right].$$

Remark. We know that $e^{i\theta} = \cos\theta + i\sin\theta$ (Euler's Theorem)

$\therefore$ $x^n \sin ax$ = Imaginary part of $x^n(\cos ax + i\sin ax)$

= I.P. of $x^n e^{iax}$

and $x^n \cos ax$ = Real part of $x^n(\cos ax + i\sin ax)$

= R.P. of $x^n e^{iax}$.

Example 13. *Solve* $(D^3 - 6D^2 + 11D - 6)y = e^{-2x} + e^{-3x}$.

Sol. A.E. is $D^3 - 6D^2 + 11D - 6 = 0$ or $(D-1)(D-2)(D-3) = 0$

whence $D = 1, 2, 3$

$\therefore$ $\text{C.F.} = c_1e^x + c_2e^{2x} + c_3e^{3x}$

$$\text{P.I.} = \frac{1}{D^3 - 6D^2 + 11D - 3}(e^{-2x} + e^{-3x})$$

$$= \frac{1}{D^3 - 6D^2 + 11D - 6}e^{-2x} + \frac{1}{D^3 - 6D^2 + 11D - 6}e^{-3x}$$

$$= \frac{1}{(-2)^3 - 6(-2)^2 + 11(-2) - 6}e^{-2x} + \frac{1}{(-3)^3 - 6(-3)^2 + 11(-3) - 6}e^{-3x}$$

$$= -\frac{1}{60}e^{-2x} - \frac{1}{120}e^{-3x} = -\frac{1}{120}(2e^{-2x} + e^{-3x})$$

Hence the C.S. is y = C.F. + P.I.

i.e., $$y = c_1e^x + c_2e^{2x} + c_3e^{3x} - \frac{1}{120}(2e^{-2x} + e^{-3x}).$$

Example 14. *Solve* $(D - 2)^2y = 8(e^{2x} + \sin 2x + x^2)$.

Sol. A.E. is $(D - 2)^2 = 0$ whence $D = 2, 2$

$\therefore \quad$ C.F. $= (c_1 + c_2x)e^{2x}$

$$\text{P.I.} = \frac{1}{(D-2)^2}\,[8(e^{2x} + \sin 2x + x^2)]$$

$$= 8\left[\frac{1}{(D-2)^2}\,e^{2x} + \frac{1}{(D-2)^2}\sin 2x + \frac{1}{(D-2)^2}\,x^2\right]$$

Now, $\dfrac{1}{(D-2)^2}\,e^{2x} = x\,.\,\dfrac{1}{2(D-2)}\,e^{2x}$ | Case of failure

$= x^2\,.\,\dfrac{1}{2}\,e^{2x}$ | Case of failure

$= \dfrac{x^2}{2}\,e^{2x}$

$$\frac{1}{(D-2)^2}\sin 2x = \frac{1}{D^2 - 4D + 4}\sin 2x = \frac{1}{-2^2 - 4D + 4}\sin 2x$$

[Putting $D^2 = -2^2$]

$$= -\frac{1}{4D}\sin 2x = -\frac{1}{4}\int \sin 2x\,dx = -\frac{1}{4}\left(-\frac{\cos 2x}{2}\right) = \frac{1}{8}\cos 2x$$

$$\frac{1}{(D-2)^2}x^2 = \frac{1}{(2-D)^2}x^2 = \frac{1}{4\left(1-\frac{D}{2}\right)^2}x^2 = \frac{1}{4}\left(1-\frac{D}{2}\right)^{-2}x^2$$

$$= \frac{1}{4}\left[1 - 2\left(-\frac{D}{2}\right) + \frac{(-2)(-3)}{2}\left(\frac{D}{2}\right)^2 \ldots\ldots\right]x^2$$

$$= \frac{1}{4}\left[1 + D + \frac{3}{4}D^2 + \ldots\ldots\right]x^2$$

$$= \frac{1}{4}\left[x^2 + D(x^2) + \frac{3}{4}D^2(x^2)\right]$$

$$\therefore \quad \text{P.I.} = 8\left[\frac{x^2}{2}e^{2x} + \frac{1}{8}\cos 2x + \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right)\right]$$

$$= 4x^2\,e^{2x} + \cos 2x + 2x^2 + 4x + 3$$

Hence the C.S. is $y = (c_1 + c_2x)\,e^{2x} + 4x^2\,e^{2x} + \cos 2x + 2x^2 + 4x + 3$.

Example 15. *Solve:* $(D + 2)(D - 1)^2\,y = e^{-2x} + 2\sinh x$.

Sol. A.E. is $(D + 2)(D - 1)^2 = 0$ so that $D = -2, 1, 1$

$\therefore \quad$ C.F. $= c_1e^{-2x} + (c_2 + c_3x)\,e^x$

$$\text{P.I.} = \frac{1}{(D+2)(D-1)^2}\,(e^{-2x} + 2\sinh x)$$

$$= \frac{1}{(D+2)(D-1)^2}\,(e^{-2x} + e^x - e^{-x}) \qquad \left[\because\ \sinh x = \frac{e^x - e^{-x}}{2}\right]$$

Now $\frac{1}{(D+2)(D-1)^2}e^{-2x} = \frac{1}{D+2}\left[\frac{1}{(D-1)^2}e^{-2x}\right] = \frac{1}{D+2}\left[\frac{1}{(-2-1)^2}e^{-2x}\right]$

$$= \frac{1}{9}\cdot\frac{1}{D+2}e^{-2x} \qquad \text{| Case of failure}$$

$$= \frac{1}{9}x\,.\,\frac{1}{1}e^{-2x} = \frac{x}{9}e^{-2x}$$

$$\frac{1}{(D+2)(D-1)^2}e^{x} = \frac{1}{(D-1)^2}\left[\frac{1}{D+2}e^{x}\right] = \frac{1}{(D-1)^2}\left[\frac{1}{1+2}e^{x}\right]$$

$$= \frac{1}{3}\cdot\frac{1}{(D-1)^2}e^{x} \qquad \text{| Case of failure}$$

$$= \frac{1}{3}\,.\,x\frac{1}{2(D-1)}e^{x} \qquad \text{| Case of failure}$$

$$= \frac{1}{3}\,.\,x^2\,.\,\frac{1}{2}e^{x} = \frac{1}{6}x^2e^{x}$$

$$\frac{1}{(D+2)(D-1)^2}e^{-x} = \frac{1}{(-1+2)(-1-1)^2}e^{-x} = \frac{1}{4}e^{-x}$$

$\therefore$ $$\text{P.I.} = \frac{x}{9}e^{-2x} + \frac{x^2}{6}e^{x} + \frac{1}{4}e^{-x}$$

Hence the C.S. is $y = c_1e^{-2x} + (c_2 + c_3x)e^{x} + \frac{x}{9}e^{-2x} + \frac{x^2}{6}e^{x} + \frac{1}{4}e^{-x}.$

Example 16. *Solve* $\frac{d^2y}{dx^2} - 4y = x \sinh x$.

Sol. Given equation in symbolic form is $(D^2 - 4)y = x \sinh x$

A.E. is $D^2 - 4 = 0$ so that $D = \pm 2$

$\therefore$ C.F. $= c_1e^{2x} + c_2e^{-2x}$

$$\text{P.I.} = \frac{1}{D^2-4}x\sinh x = \frac{1}{D^2-4}x\left(\frac{e^x-e^{-x}}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4}e^{x}\,.\,x - \frac{1}{D^2-4}e^{-x}\,.\,x\right]$$

$$= \frac{1}{2}\left[e^{x}\frac{1}{(D+1)^2-4}x - e^{-x}\frac{1}{(D-1)^2-4}x\right]$$

$$= \frac{1}{2}\left[e^{x}\frac{1}{D^2+2D-3}x - e^{-x}\frac{1}{D^2-2D-3}x\right]$$

$$= \frac{1}{2}\left[e^{x}\frac{1}{-3\left(1-\frac{2D}{3}-\frac{D^2}{3}\right)}x - e^{-x}\frac{1}{-3\left(1+\frac{2D}{3}-\frac{D^2}{3}\right)}x\right]$$

$$= -\frac{1}{6}\left[e^{x}\left\{1-\left(\frac{2D}{3}+\frac{D^2}{3}\right)\right\}^{-1}x - e^{-x}\left\{1+\left(\frac{2D}{3}-\frac{D^2}{3}\right)\right\}^{-1}x\right]$$

NOTES

$$= -\frac{1}{6}\left[e^{x}\left(1+\frac{2D}{3}......\right)x - e^{-x}\left(1-\frac{2D}{3}......\right)x\right]$$

$$= -\frac{1}{6}\left[e^{x}\left(x+\frac{2}{3}\right) - e^{-x}\left(x-\frac{2}{3}\right)\right]$$

$$= -\frac{x}{3}\left(\frac{e^{x}-e^{-x}}{2}\right) - \frac{2}{9}\left(\frac{e^{x}+e^{-x}}{2}\right) = -\frac{x}{3}\sinh x - \frac{2}{9}\cosh x$$

Hence the C.S. is $y = c_1e^{2x} + c_2e^{-2x} - \frac{x}{3}\sinh x - \frac{2}{9}\cosh x$.

Example 17. *Solve* $\frac{d^4y}{dx^4} - y = \cos x \cosh x$.

Sol. Given equation in symbolic form is $(D^4 - 1)y = \cos x \cosh x$

A.E. is $D^4 - 1 = 0$ or $(D^2 - 1)(D^2 + 1) = 0$ so that $D = \pm 1, \pm i$

$\therefore$ C.F. $= c_1e^{x} + c_2e^{-x} + e^{0x}(c_3 \cos x + c_4 \sin x)$

$= c_1e^{x} + c_2e^{-x} + c_3 \cos x + c_4 \sin x$

$$\text{P.I.} = \frac{1}{D^4-1}\cos x \cosh x = \frac{1}{D^4-1}\cos x\left(\frac{e^{x}+e^{-x}}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{D^4-1}e^{x}\cos x + \frac{1}{D^4-1}e^{-x}\cos x\right]$$

$$= \frac{1}{2}\left[e^{x}\frac{1}{(D+1)^4-1}\cos x + e^{-x}\frac{1}{(D-1)^4-1}\cos x\right]$$

$$= \frac{1}{2}\left[e^{x}\frac{1}{D^4+4D^3+6D^2+4D}\cos x + e^{-x}\frac{1}{D^4-4D^3+6D^2-4D}\cos x\right]$$

$$= \frac{1}{2}\left[e^{x}\frac{1}{(-1^2)^2+4D(-1^2)+6(-1^2)+4D}\cos x\right.$$
$$\left. + e^{-x}\frac{1}{(-1^2)^2-4D(-1^2)+6(-1^2)-4D}\cos x\right]$$

$$= \frac{1}{2}\left[e^{x}\frac{1}{-5}\cos x + e^{-x}\frac{1}{-5}\cos x\right] = -\frac{1}{5}\left(\frac{e^{x}+e^{-x}}{2}\right)\cos x$$

$$= -\frac{1}{5}\cosh x \cos x$$

Hence the C.S. is $y = c_1e^{x} + c_2e^{-x} + c_3 \cos x + c_4 \sin x - \frac{1}{5}\cos x \cosh x$.

Example 18. *Solve* $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^{x} \sin x$.

Sol. Given equation in symbolic form is $(D^2 - 2D + 1)y = xe^{x} \sin x$

A.E. is $D^2 - 2D + 1 = 0$ or $(D - 1)^2 = 0$ so that $D = 1, 1$

$\therefore$ C.F. $= (c_1 + c_2x)e^{x}$

$$\text{P.I} = \frac{1}{(D-1)^2}e^{x}\,.\,x \sin x = e^{x}\,.\,\frac{1}{(D+1-1)^2}x \sin x$$

$$= e^x \frac{1}{D^2} x \sin x = e^x \frac{1}{D} \int x \sin x \, dx \quad \text{Integrating by parts}$$

$$= e^x \frac{1}{D}\left[x(-\cos x) - \int 1(-\cos x)\,dx\right] = e^x \frac{1}{D}(-x\cos x + \sin x)$$

$$= e^x \int (-x\cos x + \sin x)\,dx = e^x\left[-\left\{x\sin x - \int 1.\sin x\,dx\right\} - \cos x\right]$$

$$= e^x[-x\sin x - \cos x - \cos x] = -e^x(x\sin x + 2\cos x)$$

Hence the C.S. is $y = (c_1 + c_2x)e^x - e^x(x\sin x + 2\cos x)$.

Example 19. *Solve* $\frac{d^2y}{dx^2} - 4y = \cosh(2x-1) + 3^x$.

Sol. Given equation in symbolic form is

$$(D^2 - 4)y = \cosh(2x-1) + 3^x$$

A.E. is $D^2 - 4 = 0 \Rightarrow D = \pm 2$

$\therefore$ C.F. $= c_1e^{2x} + c_2e^{-2x}$

$$\text{P.I.} = \frac{1}{D^2-4}[\cosh(2x-1) + 3^x]$$

$$= \frac{1}{D^2-4}\left[\frac{e^{2x-1} + e^{-(2x-1)}}{2} + e^{\log 3^x}\right] \quad \left[\because \cosh t = \frac{e^t + e^{-t}}{2} \text{ and } u = e^{\log u}\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4}e^{2x-1} + \frac{1}{D^2-4}e^{-(2x-1)}\right] + \frac{1}{D^2-4}e^{x\log 3}$$

$$= \frac{1}{2}\left[x.\frac{1}{2D}e^{2x-1} + x.\frac{1}{2D}e^{-(2x-1)}\right] + \frac{1}{(\log 3)^2 - 4}e^{x\log 3}$$

$$= \frac{1}{2}\left[x.\frac{1}{2D}e^{2x-1} + x.\frac{1}{2D}e^{-(2x-1)}\right] + \frac{1}{(\log 3)^2 - 4}e^{x\log 3}$$

$$= \frac{x}{4}\left[\int e^{2x-1}\,dx + \int e^{-(2x-1)}dx\right] + \frac{3^x}{(\log 3)^2 - 4}$$

$$= \frac{x}{4}\left[\frac{e^{2x-1}}{2} + \frac{e^{-(2x-1)}}{-2}\right] + \frac{3^x}{(\log 3)^2 - 4}$$

$$= \frac{x}{4}\left[\frac{e^{2x-1} - e^{-(2x-1)}}{2}\right] + \frac{3^x}{(\log 3)^2 - 4}$$

$$= \frac{x}{4}\sinh(2x-1) + \frac{3^x}{(\log 3)^2 - 4}$$

Hence the C.S. is $y = c_1e^{2x} + c_2e^{-2x} + \frac{x}{4}\sinh(2x-1) + \frac{3^x}{(\log 3)^2 - 4}$.

Example 20. *Solve* $(D^2 + 1)y = x^2 \sin 2x$.

Sol. A.E. is $D^2 + 1 = 0 \Rightarrow D = \pm i$

$\therefore$ C.F. $= c_1 \cos x + c_2 \sin x$

$$\text{P.I.} = \frac{1}{D^2+1}x^2\sin 2x = \text{I.P. of } \frac{1}{D^2+1}x^2e^{2ix}$$

NOTES

$$= \text{I.P. of } e^{2ix} \frac{1}{(D+2i)^2+1} x^2 = \text{I.P. of } e^{2ix} \frac{1}{D^2+4iD-3} x^2$$

$$= \text{I.P. of } e^{2ix} \frac{1}{-3\left(1-\frac{4}{3}iD-\frac{D^2}{3}\right)} x^2$$

$$= \text{I.P. of } \frac{e^{2ix}}{-3}\left[1-\left(\frac{4iD+D^2}{3}\right)\right]^{-1} x^2$$

$$= \text{I.P. of } -\frac{1}{3} e^{2ix}\left[1+\left(\frac{4iD+D^2}{3}\right)+\left(\frac{4iD+D^2}{3}\right)^2+\ldots\ldots\right] x^2$$

$$= \text{I.P. of } -\frac{1}{3} e^{2ix}\left[1+\frac{4iD}{3}+\left(\frac{1}{3}-\frac{16}{9}\right)D^2+\ldots\ldots\right] x^2$$

$$= \text{I.P. of } -\frac{1}{3} e^{2ix}\left[x^2+\frac{4i}{3}(2x)-\frac{13}{9}(2)\right]$$

$$= \text{I.P. of } -\frac{1}{3}(\cos 2x + i\sin 2x)\left[\left(x^2-\frac{26}{9}\right)+\left(\frac{8x}{3}\right)i\right]$$

$$= -\frac{1}{3}\left[\frac{8x}{3}\cos 2x+\left(x^2-\frac{26}{9}\right)\sin 2x\right]$$

$$= -\frac{1}{27}[24x\cos 2x + (9x^2-26)\sin 2x]$$

Hence the C.S. is $y = c_1 \cos x + c_2 \sin x - \frac{1}{27}[24x \cos 2x + (9x^2 - 26) \sin 2x]$.

Example 21. *Solve* $(D^4 + 2D^2 + 1)y = x^2 \cos x$.

Sol. A.E. is $(D^2 + 1)^2 = 0 \Rightarrow D = \pm i, \pm i$

$\therefore$ C.F. $= (c_1x + c_2) \cos x + (c_3 x + c_4) \sin x$

$$\text{P.I.} = \frac{1}{(D^2+1)^2} x^2 \cos x = \text{R.P. of } \frac{1}{(D^2+1)^2} x^2 (\cos x + i \sin x)$$

$$= \text{R.P. of } \frac{1}{(D^2+1)^2} x^2 e^{ix} = \text{R.P. of } e^{ix} \frac{1}{[(D+i)^2+1]^2} x^2$$

$$= \text{R.P. of } e^{ix} \frac{1}{(D^2+2iD)^2} x^2 = \text{R.P. of } e^{ix} \frac{1}{\left[2iD\left(1+\frac{D}{2i}\right)\right]^2} x^2$$

$$= \text{R.P. of } e^{ix} \frac{1}{-4D^2\left(1-\frac{iD}{2}\right)^2} x^2 = \text{R.P. of } \frac{e^{ix}}{-4}\cdot\frac{1}{D^2}\left(1-\frac{iD}{2}\right)^{-2} x^2$$

$$= \text{R.P. of } -\frac{1}{4} e^{ix} \cdot \frac{1}{D^2}\left[1+2\left(\frac{iD}{2}\right)+3\left(\frac{iD}{2}\right)^2+\ldots\ldots\right] x^2$$

$$= \text{R.P. of } -\frac{1}{4} e^{ix} \cdot \frac{1}{D^2}\left[x^2+i(2x)-\frac{3}{4}(2)\right]$$

$$= \text{R.P. of} -\frac{1}{4} e^{ix} . \frac{1}{D}\left[\frac{x^3}{3} + ix^2 - \frac{3}{2}x\right]$$

$$= \text{R.P. of} -\frac{1}{4} e^{ix} \left[\frac{x^4}{12} + i\frac{x^3}{3} - \frac{3x^2}{4}\right]$$

$$= \text{R.P. of} -\frac{1}{48} (\cos x + i \sin x) [(x^4 - 9x^2) + (4x^3)i]$$

$$= -\frac{1}{48} [(x^4 - 9x^2) \cos x - 4x^3 \sin x]$$

Hence the C.S. is

$$y = (c_1x + c_2) \cos x + (c_3x + c_4) \sin x - \frac{1}{48} [(x^4 - 9x^2) \cos x - 4x^3 \sin x].$$

Example 22. *Solve* $\frac{d^2y}{dx^2} + y = cosec\ x.$

Sol. Given equation in symbolic form is $(D^2 + 1)y = \text{cosec } x$

A.E. is $D^2 + 1 = 0 \Rightarrow D = \pm i$

$\therefore$ C.F. $= c_1 \cos x + c_2 \sin x$

$$\text{P.I.} = \frac{1}{D^2+1} \text{cosec } x = \frac{1}{(D+i)(D-i)} \text{cosec } x$$

$$= \frac{1}{2i}\left(\frac{1}{D-i} - \frac{1}{D+i}\right) \text{cosec } x \qquad \text{(Partial Fractions)}$$

$$= \frac{1}{2i}\left(\frac{1}{D-i} \text{cosec } x - \frac{1}{D+i} \text{cosec } x\right)$$

Now $\frac{1}{D-i} \text{cosec } x = e^{ix} \int \text{cosec } x\, e^{-ix}\, dx \qquad \left[\because \frac{1}{D-a} X = e^{ax} \int X e^{-ax}\, dx\right]$

$$= e^{ix} \int \text{cosec } x (\cos x - i \sin x)\, dx = e^{ix} \int (\cot x - i)\, dx$$

$$= e^{ix} (\log \sin x - ix)$$

Changing i to $-i$, we have $\frac{1}{D+i} \text{cosec } x = e^{-ix} (\log \sin x + ix)$

$\therefore$ $$\text{P.I.} = \frac{1}{2i} [e^{ix} (\log \sin x - ix) - e^{-ix} (\log \sin x + ix)]$$

$$= \log \sin x \left(\frac{e^{ix} - e^{-ix}}{2i}\right) - x\left(\frac{e^{ix} + e^{-ix}}{2}\right)$$

$$= \log \sin x . \sin x - x \cos x$$

Hence the C.S. is $y = c_1 \cos x + c_2 \sin x + \sin x \log \sin x - x \cos x.$

Example 23. *Solve* $\frac{d^2y}{dx^2} + a^2y = tan\ ax.$

Sol. Given equation in symbolic form is $(D^2 + a^2)y = \tan ax$

A.E. is $D^2 + a^2 = 0 \Rightarrow D = \pm ia$

$\therefore$ C.F. $= c_1 \cos ax + c_2 \sin ax$

$$\text{P.I.} = \frac{1}{D^2 + a^2} \tan ax = \frac{1}{(D+ia)(D-ia)} \tan ax$$

NOTES

$$= \frac{1}{2ia}\left[\frac{1}{D-ia} - \frac{1}{D+ia}\right] \tan ax \qquad \text{(Partial Fractions)}$$

$$= \frac{1}{2ia}\left[\frac{1}{D-ia} \tan ax - \frac{1}{D+ia} \tan ax\right]$$

Now $\frac{1}{D-ia} \tan ax = e^{iax} \int \tan ax \,.\, e^{-iax}\, dx$

$$= e^{iax} \int \tan ax\,(\cos ax - i \sin ax)\, dx = e^{iax} \int \left(\sin ax - i \frac{\sin^2 ax}{\cos ax}\right) dx$$

$$= e^{iax} \int \left(\sin ax - i \frac{1-\cos^2 ax}{\cos ax}\right) dx = e^{iax} \int [\sin ax - i(\sec ax - \cos ax)]\, dx$$

$$= e^{iax}\left[-\frac{\cos ax}{a} - \frac{i}{a} \log(\sec ax + \tan ax) + i \frac{\sin ax}{a}\right]$$

$$= -\frac{1}{a} e^{iax} [(\cos ax - i \sin ax) + i \log(\sec ax + \tan ax)]$$

$$= -\frac{1}{a} e^{iax} [e^{-iax} + i \log(\sec ax + \tan ax)]$$

$$= -\frac{1}{a} [1 + ie^{iax} \log(\sec ax + \tan ax)]$$

Changing i to $-i$, we have $\frac{1}{D+ia} \tan ax = -\frac{1}{a} [1 - ie^{-iax} \log(\sec ax + \tan ax)]$

$$\therefore \quad \text{P.I.} = \frac{1}{2ia}\left[\left\{-\frac{1}{a}\{1 + ie^{iax} \log(\sec ax + \tan ax)\}\right.\right.$$
$$\left.\left. + \frac{1}{a}\{1 - ie^{-iax} \log(\sec ax + \tan ax)\}\right\}\right]$$

$$= -\frac{1}{a^2} \log(\sec ax + \tan ax) \left(\frac{e^{iax} + e^{-iax}}{2}\right)$$

$$= -\frac{1}{a^2} \log(\sec ax + \tan ax) \,.\, \cos ax$$

Hence the C.S. is $y = c_1 \cos ax + c_2 \sin ax - \frac{1}{a^2} \cos ax \log(\sec ax + \tan ax)$.

EXERCISE B

Solve the following differential equations:

1. $\frac{d^3y}{dx^3} + y = 3 + 5e^x$.
2. $\frac{d^2y}{dx^2} - 4y = (1 + e^x)^2$.
3. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = -2 \cosh x$.
4. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = \sin 3x$.
5. (*i*) $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \sin 2x$. (*ii*) $\frac{d^2y}{dx^2} + \frac{dy}{dx} = \cos 2x$

6. (*i*) $\frac{d^3y}{dx^3} + y = \sin 3x - \cos^2 \frac{x}{2}$ (*ii*) $(D^3 + 1)y = 2\cos^2 x$

(*iii*) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^{2x} - \cos^2 x$ (*iv*) $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{-x} + \sin 2x$

(*v*) $(D^3 - D)z = 2y + 1 + 4\cos y + 2e^y$, where $D \equiv \frac{d}{dy}$

(*vi*) $(D^2 + D + 1)y = (1 + \sin x)^2$

7. $(D^2 - 4D + 3)y = \sin 3x \cos 2x$.

8. $(D^2 - 3D + 2)y = 6e^{-3x} + \sin 2x$.

9. $\frac{d^2y}{dx^2} + 4y = e^x + \sin 2x$.

10. $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} + 4\frac{dy}{dx} = e^{2x} + \sin 2x$.

11. $\frac{d^2y}{dx^2} - 4y = x^2 + 2x$.

12. $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 6\frac{dy}{dx} = 1 + x^2$.

13. $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x + 4$.

14. $\frac{d^2y}{dx^2} + y = e^{2x} + \cosh 2x + x^3$.

15. $(D^2 - 3D + 2)y = 2e^x \cos \frac{x}{2}$.

16. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = xe^{3x} + \sin 2x$.

17. $\frac{d^4y}{dx^4} - y = e^x \cos x$.

18. (*i*) $(D^2 - 2D)y = e^x \sin x$. (*ii*) $y'' - 2y' + 2y = x + e^x \cos x$

19. $(D^2 + 4D + 8)y = 12e^{-2x} \sin x \sin 3x$.

20. (*i*) $\frac{d^2y}{dx^2} + 2y = x^2e^{3x} + e^x \cos 2x$. (*ii*) $(D^2 + 4D + 3)y = e^{-x} \sin x + x\,e^{3x}$.

21. $(D^3 + 2D^2 + D)y = x^2e^{2x} + \sin^2 x$.

22. $(D^2 - 4D + 4)y = 8x^2 e^{2x} \sin 2x$.

23. $(D - 1)^2(D + 1)^2y = \sin^2 \frac{x}{2} + e^x + x$.

24. $\frac{d^2y}{dx^2} + 4y = x \sin x$.

25. $(D^2 - 1)y = x^2 \sin x$.

26. $\frac{d^2y}{dx^2} - 9y = x \cos 2x$.

27. $(D^2 - 1)y = x \sin x + (1 + x^2)e^x$.

28. $(D^2 - 1)y = x \sin 3x + \cos x$.

29. $\frac{d^2y}{dx^2} + a^2y = \sec ax$.

30. $\frac{d^2y}{dx^2} + 4y = 4\tan 2x$.

31. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$.

32. Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y + 37\sin 3x = 0$ and find the value of y when $x = \frac{\pi}{2}$ being given that $y = 3$, $\frac{dy}{dx} = 0$ when $x = 0$.

Answers

1. $y = c_1e^{-x} + e^{\frac{1}{2}x}\left(c_2 \cos \frac{\sqrt{3}}{2}x + c_3 \sin \frac{\sqrt{3}}{2}x\right) + 3 + \frac{5}{2}e^x$

2. $y = c_1e^{2x} + c_2e^{-2x} - \frac{1}{4} - \frac{2}{3}e^x + \frac{1}{4}xe^{2x}$

3. $y = e^{-2x}(c_1 \cos x + c_2 \sin x) - \frac{1}{10}e^x - \frac{1}{2}e^{-x}$

4. $y = e^x(c_1 \cos 2x + c_2 \sin 2x) + \frac{1}{26}(3\cos 3x - 2\sin 3x)$

5. (*i*) $y = c_1e^{-x} + c_2 \cos x + c_3 \sin x + \frac{1}{15} (2 \cos 2x - \sin 2x)$

(*ii*) $y = c_1 + c_2e^{-x} + \frac{1}{10}(\sin 2x - 2 \cos 2x)$

6. (*i*) $y = c_1e^{-x} + e^{\frac{1}{2}x}\left(c_2 \cos \frac{\sqrt{3}}{2} x + c_3 \sin \frac{\sqrt{3}}{2} x\right) + \frac{1}{730} (\sin 3x + 27 \cos 3x)$

$- \frac{1}{2} - \frac{1}{4} (\cos x - \sin x)$

(*ii*) $y = c_1e^{-x} + e^{\frac{x}{2}}\left(c_2 \cos \frac{\sqrt{3}}{2} x + c_3 \sin \frac{\sqrt{3}}{2} x\right) + 1 + \frac{1}{65} (\cos 2x - 8 \sin 2x)$

(*iii*) $y = (c_1 + c_2x)e^{-x} + \frac{1}{9} e^{2x} - \frac{1}{2} + \frac{1}{50} (3 \cos 2x - 4 \sin 2x)$

(*iv*) $y = c_1 + (c_2 + c_3x)e^{-x} - \frac{x^2}{2} e^{-x} + \frac{1}{50}(3 \cos 2x - 4 \sin 2x)$

(*v*) $z = c_1 + c_2e^{y} + c_3e^{-y} - y^2 - y - 2 \sin y + ye^{y}$

(*vi*) $y = e^{-\frac{x}{2}}\left(c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x\right) + \frac{3}{2} - 2 \cos x - \frac{1}{13} \sin 2x + \frac{3}{26} \cos 2x$

7. $y = c_1e^{x} + c_2e^{3x} + \frac{1}{884} (10 \cos 5x - 11 \sin 5x) + \frac{1}{20} (\sin x + 2 \cos x)$

8. $y = c_1e^{x} + c_2e^{2x} + \frac{3}{10} e^{-3x} + \frac{1}{20} (3 \cos 2x - \sin 2x)$

9. $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{5} e^{x} - \frac{x}{4} \cos 2x$

10. $y = c_1 + e^{x} (c_2 \cos \sqrt{3}\, x + c_3 \sin \sqrt{3}\, x) + \frac{1}{8} (e^{2x} + \sin 2x)$

11. $y = c_1e^{2x} + c_2e^{-2x} - \frac{1}{4}\left(x^2 + 2x + \frac{1}{2}\right)$

12. $y = c_1 + c_2e^{3x} + c_3e^{-2x} - \frac{1}{18}\left(x^3 - \frac{x^2}{2} + \frac{25}{6} x\right)$

13. $y = c_1 + c_2e^{-x} + \frac{x^3}{3} + 4x$

14. $y = c_1 \cos x + c_2 \sin x + \frac{1}{5} e^{2x} + \frac{1}{5} \cosh 2x + x^3 - 6x$

15. $y = c_1e^{x} + c_2e^{2x} - \frac{8}{5} e^{x}\left(2 \sin \frac{x}{2} + \cos \frac{x}{2}\right)$

16. $y = c_1e^{x} + c_2e^{2x} + \frac{1}{4} e^{3x} (2x - 3) + \frac{1}{20} (3 \cos 2x - \sin 2x)$

17. $y = c_1e^{x} + c_2e^{-x} + c_3 \cos x + c_4 \sin x - \frac{1}{5} e^{x} \cos x$

18. (*i*) $y = c_1 + c_2e^{2x} - \frac{1}{2} e^{x} \sin x$ (*ii*) $y = e^{x} (c_1 \cos x + c_2 \sin x) + \frac{1}{2}(x + 1 + xe^{x} \sin x)$

19. $y = e^{-2x} (c_1 \cos 2x + c_2 \sin 2x) + \frac{1}{2} e^{-2x} (3x \sin 2x + \cos 4x)$

20. (*i*) $y = c_1 \cos \sqrt{2}\, x + c_2 \sin \sqrt{2}\, x + \frac{e^{3x}}{11}\left(x^2 - \frac{12}{11} x + \frac{50}{121}\right) + \frac{e^{x}}{17} (4 \sin 2x - \cos 2x)$

(*ii*) $y = c_1e^{-x} + c_2e^{-3x} - \frac{1}{5}e^{-x} (\sin x + 2 \cos x) + \frac{1}{24} e^{3x}\left(x - \frac{5}{12}\right)$

21. $y = c_1 + (c_2 + c_3x)e^{-x} + \frac{1}{108}e^{2x}(6x^2 - 14x + 11) + \frac{1}{2}x + \frac{1}{100}(3\sin 2x + 4\cos 2x)$

22. $y = (c_1 + c_2x)e^{2x} - e^{2x}[4x\cos 2x + (2x^2 - 3)\sin 2x]$

23. $y = (c_1 + c_2x)e^{x} + (c_3 + c_4x)e^{-x} + \frac{1}{2} - \frac{1}{8}\cos x + \frac{x^2}{8}e^x + x$

24. $y = c_1\cos 2x + c_2\sin 2x + \frac{1}{9}(3x\sin x - 2\cos x)$

25. $y = c_1e^x + c_2e^{-x} - x\cos x - \frac{1}{2}(x^2 - 1)\sin x$

26. $y = c_1e^{3x} + c_2e^{-3x} - \frac{1}{169}(13x\cos 2x - 4\sin 2x)$

27. $y = c_1e^x + c_2e^{-x} - \frac{1}{2}(x\sin x + \cos x) + \frac{1}{12}xe^x(2x^2 - 3x + 9)$

28. $y = c_1e^x + c_2e^{-x} - \frac{1}{10}\left(x\sin 3x + \frac{3}{5}\cos 3x\right) - \frac{1}{2}\cos x$

29. $y = c_1\cos ax + c_2\sin ax + \frac{1}{a}\left(x\sin ax + \cos ax\,\frac{\log\cos ax}{a}\right)$

30. $y = c_1\cos 2x + c_2\sin 2x - \cos 2x\log(\sec 2x + \tan 2x)$

31. $y = c_1e^{-x} + c_2e^{-2x} + e^{-2x}\,.\,e^{e^x}$

32. $y = e^{-x}(c_1\cos 3x + c_2\sin 3x) + 6\cos 3x - \sin 3x\,;\; y = 1.$

METHOD OF VARIATION OF PARAMETERS TO FIND P.I.

Consider the linear equation of **second order with constant co-efficients**

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = X \qquad \ldots(1)$$

Let its C.F. be $y = c_1y_1 + c_2y_2$ so that y_1 and y_2 satisfy the equation

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = 0 \qquad \ldots(2)$$

Now, replacing c_1, c_2 (regarded as parameters) by unknown functions $u(x)$ and $v(x)$, let us assume that the P.I. of (1) is $y = uy_1 + vy_2$ ...(3)

Differentiating (3) w.r.t. x, we have $y' = uy_1' + vy_2' + u'y_1 + v'y_2 = uy_1' + vy_2'$ ...(4)

assuming that u, v satisfy the equation $u'y_1 + v'y_2 = 0$ ...(5)

Differentiating (4) w.r.t. x, we have $y'' = uy_1'' + u'y_1' + vy_2'' + v'y_2'$

Substituting the values of y, y' and y'' in (1), we get

$$(uy_1'' + u'y_1' + vy_2'' + v'y_2') + a_1(uy_1' + vy_2') + a_2(uy_1 + vy_2) = X$$

or $$u(y_1'' + a_1y_1' + a_2y_1) + v(y_2'' + a_1y_2' + a_2y_2) + u'y_1' + v'y_2' = X$$

or $$u'y_1' + v'y_2' = X \qquad \ldots(6)$$

since y_1 and y_2 satisfy (2).

Solving (5) and (6), we get $u' = \begin{vmatrix} 0 & y_2 \\ X & y_2' \end{vmatrix} \div \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = -\dfrac{y_2 X}{W}$

and $v' = \begin{vmatrix} y_1 & 0 \\ y_1' & X \end{vmatrix} \div \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \dfrac{y_1 X}{W}$

where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is called the Wronskian of y_1, y_2.

Integrating, $u = -\displaystyle\int \frac{y_2 X}{W}\, dx, \quad v = \int \frac{y_1 X}{W}\, dx$

Substituting in (3), the P.I. is known. Thus P.I. $= -y_1 \displaystyle\int \frac{y_2 X}{W}\, dx + y_2 \int \frac{y_1 X}{W}\, dx$.

Note 1. As the solution is obtained by varying the arbitrary constants c_1, c_2 of the C.F., the method is known as *variation of parameters*.

Note 2. Method of variation of parameters is to be used if instructed to do so.

SOLVED EXAMPLES

Example 24. *Apply the method of variation of parameters to solve*

$$\frac{d^2y}{dx^2} + 4y = 4 \sec^2 2x.$$

Sol. Given equation in symbolic form is $(D^2 + 4)y = 4 \sec^2 2x$

Its A.E. is $D^2 + 4 = 0$ so that $D = \pm 2i$

$\therefore$ C.F. is $y = c_1 \cos 2x + c_2 \sin 2x$

Here, $y_1 = \cos 2x$, $y_2 = \sin 2x$ and $X = 4 \sec^2 2x$

$$\therefore \quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

$$\text{P.I.} = -y_1 \int \frac{y_2 X}{W}\, dx + y_2 \int \frac{y_1 X}{W}\, dx$$

$$= -\cos 2x \int \frac{\sin 2x \,.\, 4\sec^2 2x}{2}\, dx + \sin 2x \int \frac{\cos 2x \,.\, 4 \sec^2 2x}{2}\, dx$$

$$= -2\cos 2x \int \sec 2x \tan 2x\, dx + 2 \sin 2x \int \sec 2x\, dx$$

$$= -2\cos 2x \,.\, \frac{\sec 2x}{2} + 2 \sin 2x \,.\, \frac{1}{2} \log(\sec 2x + \tan 2x)$$

$$= -1 + \sin 2x \log(\sec 2x + \tan 2x)$$

Hence the C.S. is $y = c_1 \cos 2x + c_2 \sin 2x - 1 + \sin 2x \log(\sec 2x + \tan 2x)$.

Example 25. *Solve by the method of variation of parameters:*

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \frac{e^{3x}}{x^2}.$$

Sol. Given equation in symbolic form is

$$(D^2 - 6D + 9)y = \frac{e^{3x}}{x^2}$$

Its A.E. is $(D-3)^2 = 0 \Rightarrow D = 3, 3$

$\therefore$ C.F. is $y = (c_1 + c_2x)e^{3x}$

Here, $y_1 = e^{3x}$, $y_2 = xe^{3x}$ and $X = \dfrac{e^{3x}}{x^2}$

$$\therefore \quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & (3x+1)e^{3x} \end{vmatrix} = e^{6x}$$

$$\text{P.I.} = -y_1 \int \frac{y_2 X}{W}\, dx + y_2 \int \frac{y_1 X}{W}\, dx$$

$$= -e^{3x} \int \frac{xe^{3x} \cdot \frac{e^{3x}}{x^2}}{e^{6x}}\, dx + xe^{3x} \int \frac{e^{3x} \cdot \frac{e^{3x}}{x^2}}{e^{6x}}\, dx$$

$$= -e^{3x} \int \frac{1}{x}\, dx + xe^{3x} \int \frac{1}{x^2}\, dx$$

$$= -e^{3x} \log x + xe^{3x}\left(-\frac{1}{x}\right) = -(1 + \log x)\, e^{3x}$$

Hence, C.S. is $y = (c_1 + c_2x)\, e^{3x} - (1 + \log x)\, e^{3x}$

or $y = [(c_1 - 1) + c_2x - \log x]\, e^{3x}$

or $y = [(C_1 + c_2x - \log x)\, e^{3x}$, where $C_1 = c_1 - 1$.

Example 26. *Solve by the method of variation of parameters*

$$\frac{d^2y}{dx^2} - y = e^{-x} \sin (e^{-x}) + \cos (e^{-x}).$$

Sol. Given equation in symbolic form is

$$(D^2 - 1)y = e^{-x} \sin (e^{-x}) + \cos (e^{-x})$$

Its A.E. is $D^2 - 1 = 0 \Rightarrow D = \pm 1$

$\therefore$ C.F. is $y = c_1e^{x} + c_2e^{-x}$

Here, $y_1 = e^{x}$, $y_2 = e^{-x}$ and $X = e^{-x} \sin (e^{-x}) + \cos (e^{-x})$

$$\therefore \quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix} = -2$$

$$\text{P.I.} = -y_1 \int \frac{y_2 X}{W}\, dx + y_2 \int \frac{y_1 X}{W}\, dx$$

$$= -e^{x} \int \frac{e^{-x}\, [e^{-x} \sin (e^{-x}) + \cos (e^{-x})]}{-2}\, dx + e^{-x} \int \frac{e^{x}\, [e^{-x} \sin (e^{-x}) + \cos (e^{-x})]}{-2}\, dx$$

$$= \frac{1}{2} e^{x} \int e^{-x}\, [e^{-x} \sin (e^{-x}) + \cos (e^{-x})]\, dx - \frac{1}{2} e^{-x} \int e^{x}\, [e^{-x} \sin (e^{-x}) + \cos (e^{-x})]\, dx \quad ...(1)$$

Now, $\int e^{-x} [e^{-x} \sin (e^{-x}) + \cos (e^{-x})]\, dx$

$= -\int (t \sin t + \cos t)\, dt$, where $t = e^{-x}$

$= -[t(-\cos t) - \int 1 . (-\cos t)\, dt + \sin t]$

$= -(-t \cos t + 2 \sin t) = e^{-x} \cos (e^{-x}) - 2 \sin (e^{-x})$

Also, $\int e^{x} [\cos (e^{-x}) + e^{-x} \sin (e^{-x})]\, dx$ | Form $\int e^{x} [f(x) + f'(x)]\, dx = e^{x} f(x)$

$= e^{x} \cos (e^{-x})$

$\therefore$ From (1), we have

$$\text{P.I.} = \frac{1}{2} e^{x} [e^{-x} \cos (e^{-x}) - 2 \sin (e^{-x})] - \frac{1}{2} e^{-x} . e^{x} \cos (e^{-x})$$

$$= \frac{1}{2} \cos (e^{-x}) - e^{x} \sin (e^{-x}) - \frac{1}{2} \cos (e^{-x}) = -e^{x} \sin (e^{-x})$$

Hence, C.S. is $y = c_1 e^{x} + c_2 e^{-x} - e^{x} \sin (e^{-x})$.

EXERCISE C

Solve by the method of variation of parameters:

1. $\frac{d^2y}{dx^2} + y = \text{cosec } x.$
2. (i) $\frac{d^2y}{dx^2} + 16y = 32 \sec 2x$ (ii) $\frac{d^2y}{dx^2} + a^2 y = \sec ax$
   (iii) $y'' + y = \sec^2 x$ (iv) $y'' + 3y' + 2y = \sin (e^{x})$
3. $\frac{d^2y}{dx^2} + y = \tan x.$
4. $\frac{d^2y}{dx^2} + 4y = \tan 2x.$
5. (i) $\frac{d^2y}{dx^2} + y = x \sin x.$ (ii) $(D^2 + 1)y = \text{cosec } x \cot x$
6. (i) $y'' - 2y' + 2y = e^{x} \tan x.$ (ii) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^{x} \sin x.$
7. $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = \frac{1}{x^3} e^{-3x}.$
8. $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = \frac{12e^{4x}}{x^4}.$
9. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^{2x} \sec^2 x.$
10. $y'' - 2y' + y = e^{x} \log x.$
11. $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^{x}}.$
12. $\frac{d^2y}{dx^2} + y = \frac{1}{1+\sin x}.$

Answers

1. $y = c_1 \cos x + c_2 \sin x - x \cos x + \sin x \log \sin x$
2. (i) $y = c_1 \cos 4x + c_2 \sin 4x + 8 \cos 2x - 4 \sin 4x \log (\sec 2x + \tan 2x)$

   (ii) $y = c_1 \cos ax + c_2 \sin ax + \frac{1}{a^2} \cos ax \log (\cos ax) + \frac{1}{a} x \sin ax$

   (iii) $y = c_1 \cos x + c_2 \sin x - 1 + \sin x \log (\sec x + \tan x)$

   (iv) $y = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x} \sin (e^{x})$

3. $y = c_1 \cos x + c_2 \sin x - \cos x \log (\sec x + \tan x)$

4. $y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4} \cos 2x \log (\sec 2x + \tan 2x)$

5. (*i*) $y = c_1 \cos x + c_2 \sin x + \frac{x}{4} \sin x - \frac{x^2}{4} \cos x$

(*ii*) $y = c_1 \cos x + c_2 \sin x + \cos x \log \sin x - x \sin x$

6. (*i*) $y = e^x (c_1 \cos x + c_2 \sin x) - e^x \cos x \log (\sec x + \tan x)$

(*ii*) $y_1 = c_1 + c_2 e^{2x} - \frac{1}{2} e^x \sin x$

7. $y = \left(c_1 + c_2 x + \frac{1}{2x}\right) e^{-3x}$ **8.** $y = \left(c_1 + c_2 x + \frac{2}{x^2}\right) e^{4x}$

9. $y = (c_1 + c_2 x - \log \cos x)\, e^{2x}$ **10.** $y = (c_1 + c_2 x)\, e^x + \frac{1}{4} x^2 e^x (2 \log x - 3)$

11. $y = c_1 e^x + c_2 e^{-x} - 1 - xe^x + (e^x - e^{-x}) \log (1 + e^x)$

12. $y = c_1 \cos x + c_2 \sin x + \sin x \log (1 + \sin x) - x \cos x - 1.$

HOMOGENEOUS LINEAR EQUATIONS (Cauchy-Euler Equations)

An equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots\cdots + a_{n-1} x \frac{dy}{dx} + a_n y = X \quad ...(i)$$

where a_i's are constants and X is a function of x, is called Cauchy's homogeneous linear equation.

Such equations can be reduced to linear differential equations with constant co-efficients by the substitution $x = e^z$ or $z = \log x$

so that
$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz} \cdot \frac{1}{x} \quad \text{or} \quad x \frac{dy}{dx} = \frac{dy}{dz} = \text{D}y, \text{ where D} = \frac{d}{dz}$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x} \cdot \frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x} \cdot \frac{d^2 y}{dz^2} \cdot \frac{dz}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2 y}{dz^2} \qquad \left(\because \frac{dz}{dx} = \frac{1}{x}\right)$$

or
$$x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dz^2} - \frac{dy}{dz} = \text{D}^2 y - \text{D}y = \text{D}(\text{D} - 1)y$$

Similarly, $x^3 \frac{d^3 y}{dx^3} = \text{D}(\text{D} - 1)(\text{D} - 2)y$ and so on.

Substituting these values in equation (*i*), we get a linear differential equation with constant co-efficients, which can be solved by the methods already discussed.

SOLVED EXAMPLES

NOTES

Example 27. *Solve* $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$.

Sol. Given equation is a Cauchy's homogeneous linear equation.

Put $x = e^z$ *i.e.,* $z = \log x$

so that $$x\frac{dy}{dx} = Dy,\ x^2\frac{d^2y}{dx^2} = D(D-1)y$$

$$x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y, \text{ where } D = \frac{d}{dz}$$

Substituting these values in the given equation, it reduces to

$$[D(D-1)(D-2) + 2D(D-1) + 2]y = 10(e^z + e^{-z})$$

or $$(D^3 - D^2 + 2)y = 10(e^z + e^{-z})$$

which is a linear equation with constant co-efficients.

Its A.E. is $D^3 - D^2 + 2 = 0$ or $(D+1)(D^2 - 2D + 2) = 0$

$$\therefore \quad D = -1, \frac{2 \pm \sqrt{4-8}}{2} = -1,\ 1 \pm i$$

$$\therefore \quad \text{C.F.} = c_1e^{-z} + e^z(c_2\cos z + c_3 \sin z) = \frac{c_1}{x} + x[c_2\cos(\log x) + c_3\sin(\log x)]$$

$$\text{P.I.} = 10\frac{1}{D^3 - D^2 + 2}(e^z + e^{-z}) = 10\left(\frac{1}{D^3 - D^2 + 2}e^z + \frac{1}{D^3 - D^2 + 2}e^{-z}\right)$$

$$= 10\left(\frac{1}{1^3 - 1^2 + 2}e^z + z\,.\,\frac{1}{3D^2 - 2D}e^{-z}\right)$$

$$= 10\left(\frac{1}{2}e^z + z\,.\,\frac{1}{3(-1)^2 - 2(-1)}e^{-z}\right)$$

$$= 5e^z + 2ze^{-z} = 5x + \frac{2}{x}\log x$$

Hence the C.S. is $y = \frac{c_1}{x} + x[c_2\cos(\log x) + c_3\sin(\log x)] + 5x + \frac{2}{x}\log x$.

Example 28. *Solve* $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2 \log x$.

Sol. Given equation is a Cauchy's homogeneous linear equation.

Put $x = e^z$ *i.e.,* $z = \log x$ so that

$$x\frac{dy}{dx} = Dy,\ x^2\frac{d^2y}{dx^2} = D(D-1)y, \text{ where } D = \frac{d}{dz}.$$

Substituting these values in the given equation, it reduces to

$$[D(D-1) - D - 3]y = ze^{2z} \quad \text{or} \quad (D^2 - 2D - 3)y = ze^{2z}$$

which is a linear equation with constant co-efficients.

Its A.E. is $D^2 - 2D - 3 = 0$ or $(D-3)(D+1) = 0$

$$\therefore \quad D = 3, -1$$

$$\text{C.F.} = c_1e^{3z} + c_2e^{-z} = c_1x^3 + \frac{c_2}{x}$$

$$\text{P.I.} = \frac{1}{D^2 - 2D - 3} (e^{2z} \cdot z)$$

$$= e^{2z} \frac{1}{(D+2)^2 - 2(D+2) - 3} z = e^{2z} \frac{1}{D^2 + 2D - 3} z$$

$$= e^{2z} \frac{1}{-3\left(1 - \frac{2D}{3} - \frac{D^2}{3}\right)} z = -\frac{1}{3} e^{2z} \left[1 - \left(\frac{2D}{3} + \frac{D^2}{3}\right)\right]^{-1} z$$

$$= -\frac{1}{3} e^{2z} \left[1 + \left(\frac{2D}{3} + \frac{D^2}{3}\right) + \ldots\ldots\right] z$$

$$= -\frac{1}{3} e^{2z} \left(z + \frac{2}{3}\right) = -\frac{x^2}{3} \left(\log x + \frac{2}{3}\right)$$

Hence the C.S. is $y = c_1 x^3 + \frac{c_2}{x} - \frac{x^2}{3} \left(\log x + \frac{2}{3}\right)$.

Example 29. *Solve* $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = \log x \sin (\log x)$.

Sol. Given equation is a Cauchy's homogeneous linear equation.

Put $x = e^z$ *i.e.,* $z = \log x$ so that $x \frac{dy}{dx} = Dy,\ x^2 \frac{d^2 y}{dx^2} = D(D-1)y$

where $D = \frac{d}{dz}$.

Substituting these values in the given equation, it reduces to

$$[D(D-1) + D + 1]y = z \sin z$$

or

$$(D^2 + 1)y = z \sin z$$

Its A.E. is $D^2 + 1 = 0$ so that $D = \pm i$

$$\text{C.F.} = c_1 \cos z + c_2 \sin z = c_1 \cos (\log x) + c_2 \sin (\log x)$$

$$\text{P.I.} = \frac{1}{D^2 + 1} z \sin z = \text{Imaginary part of } \frac{1}{D^2 + 1} z e^{iz}$$

$$= \text{I.P. of } e^{iz} \frac{1}{(D+i)^2 + 1} z = \text{I.P. of } e^{iz} \frac{1}{D^2 + 2iD} z$$

$$= \text{I.P. of } e^{iz} \frac{1}{2iD\left(1 + \frac{D}{2i}\right)} z = \text{I.P of } e^{iz} \frac{1}{2iD\left(1 - \frac{iD}{2}\right)} z$$

$$= \text{I.P. of } \frac{1}{2i} e^{iz} \frac{1}{D} \left(1 - \frac{iD}{2}\right)^{-1} z = \text{I.P. of } \frac{1}{2i} e^{iz} \frac{1}{D} \left(1 + \frac{iD}{2} + \ldots\ldots\right) z$$

$$= \text{I.P. of } \frac{1}{2i} e^{iz} \frac{1}{D} \left(z + \frac{i}{2}\right) = \text{I.P. of } \frac{1}{2i} e^{iz} \int \left(z + \frac{i}{2}\right) dz$$

$$= \text{I.P of } -\frac{i}{2} e^{iz} \left(\frac{z^2}{2} + \frac{i}{2} z\right) = \text{I.P. of } e^{iz} \left(-\frac{i}{4} z^2 + \frac{z}{4}\right)$$

$$= \text{I.P. of } (\cos z + i \sin z) \left(-\frac{i}{4} z^2 + \frac{z}{4}\right) = -\frac{z^2}{4} \cos z + \frac{z}{4} \sin z$$

$$= -\frac{1}{4} (\log x)^2 \cos (\log x) + \frac{1}{4} \log x \sin (\log x)$$

NOTES

Hence the C.S. is

$$y = c_1 \cos(\log x) + c_2 \sin(\log x) - \tfrac{1}{4}(\log x)^2 \cos(\log x) + \tfrac{1}{4}\log x \sin(\log x).$$

Example 30. *Solve:* $x^2 \dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = e^x$.

Sol. Given equation is a Cauchy's homogeneous linear equation.

Put $x = e^z$ *i.e.,* $z = \log x$ so that

$$x\frac{dy}{dx} = \text{D}y,\ x^2\frac{d^2y}{dx^2} = \text{D}(\text{D}-1)y, \quad \text{where D} = \frac{d}{dz}$$

Substituting these values in the given equation, it reduces to

$$[\text{D}(\text{D}-1) + 4\text{D} + 2]y = e^{e^z} \quad \text{or} \quad (\text{D}^2 + 3\text{D} + 2)y = e^{e^z}$$

Its A.E. is $\quad \text{D}^2 + 3\text{D} + 2 = 0 \quad$ or $\quad (\text{D}+1)(\text{D}+2) = 0$

$\therefore \quad \text{D} = -1, -2$

$$\text{C.F.} = c_1e^{-z} + c_2e^{-2z} = c_1x^{-1} + c_2x^{-2}$$

$$\text{P.I.} = \frac{1}{\text{D}^2 + 3\text{D} + 2}e^{e^z} = \frac{1}{(\text{D}+1)(\text{D}+2)}e^{e^z}$$

$$= \left(\frac{1}{\text{D}+1} - \frac{1}{\text{D}+2}\right)e^{e^z} = \frac{1}{\text{D}-(-1)}e^{e^z} - \frac{1}{\text{D}-(-2)}e^{e^z}$$

$$= e^{-z}\int e^{e^z}.e^z\,dz - e^{-2z}\int e^{e^z}.e^{2z}dz \qquad \left[\because \frac{1}{\text{D}-a}\text{X} = e^{ax}\int \text{X}\cdot e^{-ax}\,dx\right]$$

$$= e^{-z}\int e^{e^z}.e^z\,dz - e^{-2z}\int e^{e^z}.e^z.e^z\,dz \qquad |\ \text{Put } e^z = t$$

$$= e^{-z}\int e^t\,dt - e^{-2z}\int te^t\,dt$$

$$= e^{-z}.e^t - e^{-2z}(t-1)e^t \qquad |\ \text{Integrating by parts}$$

$$= e^{-z}.e^{e^z} - e^{-2z}(e^z - 1)e^{e^z}$$

$$= (e^{-z} - e^{-z} + e^{-2z})e^{e^z} = e^{-2z}.e^{e^z}$$

$$= x^{-2}e^x$$

Hence the C.S. is $y = c_1x^{-1} + c_2x^{-2} + x^{-2}e^x$ or $y = (c_1x + c_2 + e^x)x^{-2}$.

LEGENDRE'S LINEAR DIFFERENTIAL EQUATION

An equation of the form

$$(a+bx)^n\frac{d^ny}{dx^n} + a_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots\cdots + a_{n-1}(a+bx)\frac{dy}{dx} + a_ny = \text{X} \qquad \ldots(i)$$

where a_i's are constants and X is a function of x, is called Legendre's linear equation.

Such equations can be reduced to linear differential equations with constant co-efficients, by the substitution $a + bx = e^z$ *i.e.,* $z = \log(a+bx)$ so that

$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{b}{a+bx}\frac{dy}{dz}$$

or $\qquad (a+bx)\dfrac{dy}{dx} = b\dfrac{dy}{dz} = b\,\text{D}y, \quad \text{where D} = \dfrac{d}{dz}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{b}{a+bx}\frac{dy}{dz}\right) = -\frac{b^2}{(a+bx)^2}\frac{dy}{dz} + \frac{b}{a+bx}\cdot\frac{d^2y}{dz^2}\cdot\frac{dz}{dx}$$

$$= -\frac{b^2}{(a+bx)^2}\frac{dy}{dz} + \frac{b}{a+bx}\frac{d^2y}{dz^2}\cdot\frac{b}{a+bx} = \frac{b^2}{(a+bx)^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

or $\quad (a+bx)^2 \dfrac{d^2y}{dx^2} = b^2 (D^2y - Dy) = b^2 D(D-1)y$

Similarly, $(a+bx)^3 \dfrac{d^3y}{dx^3} = b^3 D(D-1)(D-2)y.$

Substituting these values in equation (*i*), we get a linear differential equation with constant co-efficients, which can be solved by the methods already discussed.

SOLVED EXAMPLES

Example 31. *Solve* $(3x+2)^2 \dfrac{d^2y}{dx^2} + 3(3x+2)\dfrac{dy}{dx} - 36y = 3x^2 + 4x + 1.$

Sol. Given equation is a Legendre's linear equation.

Put $\quad 3x + 2 = e^z \quad$ *i.e.*, $\quad z = \log(3x+2) \quad$ so that $\quad (3x+2)\dfrac{dy}{dx} = 3Dy,$

$$(3x+2)^2 \frac{d^2y}{dx^2} = 3^2 D(D-1)y, \quad \text{where } D = \frac{d}{dz}.$$

Substituting these values in the given equation, it reduces to

$$[3^2 D(D-1) + 3.3D - 36]y = 3\left(\frac{e^z-2}{3}\right)^2 + 4\left(\frac{e^z-2}{3}\right) + 1$$

or $$9(D^2-4)y = \frac{1}{3}e^{2z} - \frac{1}{3} \quad \text{or} \quad (D^2-4)y = \frac{1}{27}(e^{2z}-1)$$

which is a linear equation with constant co-efficients.

Its A.E. is $\quad D^2 - 4 = 0 \quad \therefore \quad D = \pm 2$

$$\text{C.F.} = c_1e^{2z} + c_2e^{-2z} = c_1(3x+2)^2 + c_2(3x+2)^{-2}$$

$$\text{P.I.} = \frac{1}{27}\cdot\frac{1}{D^2-4}(e^{2z}-1) = \frac{1}{27}\left[\frac{1}{D^2-4}e^{2z} - \frac{1}{D^2-4}e^{0z}\right]$$

$$= \frac{1}{27}\left[z\cdot\frac{1}{2D}e^{2z} - \frac{1}{0-4}e^{0z}\right] = \frac{1}{27}\left[\frac{z}{2}\int e^{2z}\,dz + \frac{1}{4}\right]$$

$$= \frac{1}{27}\left[\frac{z}{4}e^{2z} + \frac{1}{4}\right] = \frac{1}{108}(ze^{2z}+1) = \frac{1}{108}[(3x+2)^2\log(3x+2) + 1]$$

Hence the C.S. is

$$y = c_1(3x+2)^2 + c_2(3x+2)^{-2} + \frac{1}{108}[(3x+2)^2\log(3x+2) + 1].$$

NOTES

EXERCISE D

Solve:

1. (*i*) $x^2 y'' + 4xy' + 2y = 0$. (*ii*) $x^2 \frac{d^2y}{dx^2} + 9x \frac{dy}{dx} + 25y = 50$.

2. $x^2 \frac{d^2y}{dx^2} - 2y = x^2 + \frac{1}{x}$.

3. $x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - 20y = (x+1)^2$.

4. $x^2 \frac{d^3y}{dx^3} - 4x \frac{d^2y}{dx^2} + 6 \frac{dy}{dx} = 4$. [**Hint.** Multiply throughout by x]

5. (*i*) $x^4 \frac{d^3y}{dx^3} + 2x^3 \frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} + xy = 1$. (*ii*) $x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + 6y = x^2$.

(*iii*) $x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + y = \frac{1}{(1-x)^2}$.

6. The radial displacement u in a rotating disc at a distance r from the axis is given by $r^2 \frac{d^2u}{dr^2} + r \frac{du}{dr} - u + kr^3 = 0$, where k is a constant. Solve the equation under the conditions $u = 0$ when $r = 0$, $u = 0$ when $r = a$.

7. $x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + y = \log x$.

8. $x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + 2y = x \log x$.

9. $x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - 12y = x^3 \log x$.

10. $x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} - 4y = x^2 + 2 \log x$.

11. $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 5y = \sin(\log x)$.

12. $x^3 \frac{d^3y}{dx^3} + 3x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + 8y = 65 \cos(\log x)$.

13. $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 5y = x^2 \sin(\log x)$.

14. $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + y = \log x \frac{\sin(\log x) + 1}{x}$.

15. (*i*) $\frac{d^2y}{dx^2} + \frac{1}{x} \cdot \frac{dy}{dx} = \frac{12 \log x}{x^2}$. (*ii*) $x^2y'' - 4xy' + 8y = 4x^3 + 2 \sin(\log x)$

16. (*i*) $(1+x)^2 \frac{d^2y}{dx^2} + (1+x) \frac{dy}{dx} + y = 4 \cos \log(1+x)$.

(*ii*) $(1+x)^2 \frac{d^2y}{dx^2} + (1+x) \frac{dy}{dx} + y = 2 \sin[\log(1+x)]$

(*iii*) $(1+x)^2 \frac{d^2y}{dx^2} + (1+x) \frac{dy}{dx} + y = \sin[2 \log(1+x)]$

17. $(x+1)^2 \frac{d^2y}{dx^2} + (x+1) \frac{dy}{dx} = (2x+3)(2x+4)$

18. $(1+2x)^2 \frac{d^2y}{dx^2} - 6(1+2x) \frac{dy}{dx} + 16y = 8(1+2x)^2$

19. $(2x+3)^2 \frac{d^2y}{dx^2} - 2(2x+3) \frac{dy}{dx} - 12y = 6x$.

Answers

1. (*i*) $y = c_1x^{-1} + c_2x^{-2}$ (*ii*) $y = x^{-4}[c_1 \cos(3 \log x) + c_2 \sin(3 \log x)] + 2$

2. $y = c_1x^2 + \frac{c_2}{x} + \frac{1}{3}\left(x^2 - \frac{1}{x}\right) \log x$

3. $y = c_1x^{-5} + c_2x^{-4} - \frac{x^2}{14} - \frac{x}{9} - \frac{1}{20}$

4. $y = c_1 + c_2x^3 + c_3x^4 + \frac{2}{3} x$

5. (*i*) $y = (c_1 + c_2 \log x)x + c_2x^{-1} + \frac{1}{4x} \log x$

(*ii*) $y = c_1x^2 + c_2x^3 - x^2 \log x$ (*iii*) $y = \frac{1}{x}(c_1 + c_2 \log x) + \frac{1}{x} \log \frac{x}{1-x}$

6. $u = \frac{kr}{8}(a^2 - r^2)$

7. $y = (c_1 + c_2 \log x)x + \log x + 2$

8. $y = x[c_1 \cos(\log x) + c_2 \sin(\log x)] + x \log x$

9. $y = c_1x^3 + c_2x^{-4} + \frac{x^3}{98} \log x\,(7 \log x - 2)$

10. $y = c_1x^{-1} + c_2x^4 - \frac{x^2}{6} - \frac{1}{2}\log x + \frac{3}{8}$

11. $y = x^2[c_1 \cos(\log x) + c_2 \sin(\log x)] + \frac{1}{8}[\sin(\log x) + \cos(\log x)]$

12. $y = c_1x^{-2} + x[c_2 \cos(\sqrt{3}\log x) + c_3 \sin(\sqrt{3}\log x)] + 8\cos(\log x) - \sin(\log x)$

13. $y = x^2[c_1 \cos(\log x) + c_2 \sin(\log x)] - \frac{1}{2}x^2 \log x \cos(\log x)$

14. $y = c_1x^{2+\sqrt{3}} + c_2x^{2-\sqrt{3}} + \frac{1}{61x}\left[\log x\,\{5\sin(\log x) + 6\cos(\log x)\} + \frac{2}{61}\{27\sin(\log x) + 191\cos(\log x)\}\right] + \frac{1}{6x}(1 + \log x)$

15. (*i*) $y = c_1 + c_2 \log x + 2(\log x)^3$

(*ii*) $y = x^{5/2}\left[c_1 \cos\left(\frac{\sqrt{7}}{2}\log x\right) + c_2 \sin\left(\frac{\sqrt{7}}{2}\log x\right) + 2x^3 + \frac{5}{37}\cos(\log x) + \frac{7}{37}\sin(\log x)\right]$

16. (*i*) $y = c_1 \cos[\log(1+x)] + c_2 \sin[\log(1+x)] + 2\log(1+x)\sin[\log(1+x)]$

(*ii*) $y = c_1 \cos[\log(1+x)] + c_2 \sin[\log(1+x)] - \log(1+x)\cos[\log(1+x)]$

(*iii*) $y = c_1 \cos[\log(1+x)] + c_2 \sin[\log(1+x)] - \frac{1}{3}\sin[2\log(1+x)]$

17. $y = c_1 + c_2 \log(x+1) + [\log(x+1)]^2 + x^2 + 8x$

18. $y = (1+2x)^2[c_1 + c_2 \log(1+2x) + \{\log(1+2x)\}^2]$

19. $y = c_1(2x+3)^{-1} + c_2(2x+3)^3 - \frac{3}{16}(2x+3) + \frac{3}{4}$.

LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER

The general form of linear equation of second order may be written as

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

where P, Q and R are the functions of x only. There is no general method of solving this type of equations, but we will consider some particular cases in which the integral can be found.

COMPLETE SOLUTION IN TERMS OF KNOWN INTEGRAL

If an integral included in the complementary function of a linear equation of second order be known then the complete solution can be found. Let $y = u$ be an integral in the complementary function of the equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad \ldots(1)$$

NOTES

Then put, $y = uv$ so that $\frac{dy}{dx} = u_1 v + u v_1$

and $\frac{d^2 y}{dx^2} = u_2 v + 2u_1 v_1 + u v_2$

Putting in (1), we get

$$(u_2 v + 2u_1 v_1 + u v_2) + P(u_1 v + u v_1) + Quv = R$$

or $$u v_2 + (2u_1 + Pu) v_1 + (u_2 + Pu_1 + Qu) v = R$$

Since $y = u$ is a solution of

$$\frac{d^2 y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \quad \therefore \quad u_2 + Pu_1 + Qu = 0$$

So, we have $u v_2 + (2u_1 + Pu) v_1 = R$ or $v_2 + \left(\frac{2}{u} u_1 + P\right) v_1 = \frac{R}{u}$

Putting $v_1 = p$, so that $v_2 = \frac{dp}{dx}$, we get

$$\frac{dp}{dx} + \left(\frac{2}{u} . u_1 + P\right) p = \frac{R}{u} \qquad ...(2)$$

which is a linear equation in p.

$$\therefore \quad \text{I.F.} = e^{\int \left(\frac{2}{u} . u_1 + P\right) dx} = e^{\int \frac{2}{u} du + \int P dx}$$

$$= e^{2 \log u + \int P\, dx} = u^2 e^{\int P\, dx}$$

$\therefore$ We have $$p . u^2 e^{\int P\, dx} = \int \left(\frac{R}{u} . u^2 e^{\int P\, dx}\right) dx + c_1$$

$$\therefore \quad p = u^{-2} e^{-\int P\, dx} \int \left(R u e^{\int P\, dx}\right) dx + c_1 u^{-2} e^{-\int P\, dx}$$

or $$v_1 = \frac{dv}{dx} = u^{-2} e^{-\int P\, dx} \int \left(R u e^{\int P\, dx}\right) dx + c_1 u^{-2} e^{-\int P\, dx}$$

Integrating again, we have

$$v = \int \left(u^{-2} e^{-\int P\, dx} . \int R u\, e^{\int P\, dx}\, dx\right) dx + c_1 \int \left(u^{-2} e^{-\int P\, dx}\right) dx + c_2$$

$\therefore$ The complete solution of equation (1) is

$$y = uv = u \int \left(u^{-2} e^{-\int P\, dx} . \int R u\, e^{\int P\, dx}\, dx\right) dx + c_1 u \int \left(u^{-2} e^{-\int P\, dx}\right) dx + c_2 u$$

The above solution contains only two arbitrary constants.

TO FIND A PARTICULAR INTEGRAL OF $\frac{d^2 y}{dx^2} + P\frac{dy}{dx} + QY = 0$

$y = e^{mx}$ is a solution

If $y = e^{mx}$

Then $\frac{dy}{dx} = m e^{mx}$ and $\frac{d^2 y}{dx^2} = m^2 e^{mx}$

$\therefore$ If $y = e^{mx}$ is a solution of (1), then

$$(m^2 + Pm + Q)e^{mx} = 0 \quad \text{or} \quad \mathbf{m^2 + Pm + Q = 0.}$$

Deduction. (*i*) $y = e^x$ is a solution of (1), if **$1 + P + Q = 0$.**

(*ii*) $y = e^{-x}$ is the solution of (1), if $1 - P + Q = 0$

(*iii*) $y = e^{ax}$ is the solution of (1), if $a^2 + Pa + Q = 0$ or $1 + \dfrac{P}{a} + \dfrac{Q}{a^2} = 0.$

$y = x^m$ is a solution

If $\qquad y = x^m$

Then $\qquad \dfrac{dy}{dx} = mx^{m-1}$ and $\dfrac{d^2y}{dx^2} = m(m-1)x^{m-2}$

$\therefore$ If $y = x^m$ is a solution of (1), then $m(m-1)x^{m-2} + Pmx^{m-1} + Qx^m = 0$

or
$$m(m-1) + Pmx + Qx^2 = 0.$$

Deduction. (*i*) $y = x$ is the solution of (1), if $P + Qx = 0$.

(*ii*) $y = x^2$ is the solution of (1), if $2 + 2Px + Qx^2 = 0$.

Note. One integral belonging to the complementary function can be found by inspection. For this following rules are observed :

(*i*) **$y = x$ is a part of C.R., if $P + Qx = 0$**

(*ii*) **$y = e^x$ is a part of C.F., if $1 + P + Q = 0$ (i.e., sum of the co-efficients are zero)**

(*iii*) **$y = e^{-x}$ is a part of C.F., if $1 - P + Q = 0$**

(*iv*) **$y = e^{ax}$ is a part of C.F., if $1 + \dfrac{P}{a} + \dfrac{Q}{a^2} = 0$**

(*v*) **$y = x^2$ is part of C.F., if $2 + 2Px + Qx^2 = 0$.**

SOLVED EXAMPLES

Example 32. *Solve:* $\quad x^2 \dfrac{d^2y}{dx^2} - 2x(1+x)\dfrac{dy}{dx} + 2(1+x)y = x^3.$

Sol. The given equation can be written as

$$\frac{d^2y}{dx^2} - 2\left(\frac{1}{x}+1\right)\frac{dy}{dx} + 2\left(\frac{1}{x^2}+\frac{1}{x}\right)y = x$$

where
$$P + Qx = -2\left(\frac{1}{x}+1\right) + 2x\left(\frac{1}{x^2}+\frac{1}{x}\right) = 0$$

$\therefore \qquad y = x$ is a part of C.F.

Putting $y = vx$ so that

$$\frac{dy}{dx} = \frac{dv}{dx}x + v$$

and
$$\frac{d^2y}{dx^2} = \frac{d^2v}{dx^2} + 2\frac{dv}{dx}, \text{ we get}$$

$$\frac{d^2v}{dx^2} - 2\frac{dv}{dx} = 1 \qquad \text{or} \qquad \frac{dp}{dx} - 2p = 1 \qquad \text{where} \quad p = \frac{dv}{dx}$$

NOTES

which is a linear equation

$$\text{I.F.} = e^{-2\int dx} = e^{-2x}$$

$$\therefore \quad pe^{-2x} = \int 1 \,.\, e^{-2x}\, dx + c_1 = -\frac{1}{2} e^{-2x} + c_1$$

$$\therefore \quad p = \frac{dv}{dx} = -\frac{1}{2} + c_1 e^{-2x}$$

Integrating, we get $v = -\frac{1}{2}x + \frac{c_1}{2}e^{2x} + c_2$

$\therefore$ The complete solution is

$$y = vx = -\frac{1}{2}x^2 + \frac{c_1}{2}xe^{2x} + c_2 x.$$

Example 33. *Solve:* $x^2 \frac{d^2y}{dx^2} - (x^2 + 2x)\frac{dy}{dx} + (x + 2)y = x^3 e^x.$

Sol. The given equation can be written as

$$\frac{d^2y}{dx^2} - \left(1 + \frac{2}{x}\right)\frac{dy}{dx} + \left(\frac{1}{x} + \frac{2}{x^2}\right) y = xe^x$$

Here $\quad P = -\left(1 + \frac{2}{x}\right), \quad Q = \frac{1}{x} + \frac{2}{x^2} \quad$ and $\quad R = xe^x$

Since $P + Qx = 0$

$\therefore \quad y = x$ is a part of the C.F.

Putting $y = vx$, so that

$$\frac{dy}{dx} = \frac{dv}{dx} \,.\, x + v \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{d^2v}{dx^2}x + 2\frac{dv}{dx}$$

We get $\quad \frac{d^2v}{dx^2} - \frac{dv}{dx} = e^x \quad$ or $\quad \frac{dp}{dx} - p = e^x, \quad$ where $p = \frac{dv}{dx}$

which is a linear equation,

$$\text{I.F.} = e^{-\int dx} = e^{-x}$$

$$\therefore \quad pe^{-x} = \int e^{-x} \,.\, e^x\, dx + c_1 = x + c_1$$

$$\therefore \quad p = \frac{dv}{dx} = xe^x + c_1 e^x$$

Integrating, we get $v = xe^x - e^x + c_1 e^x + c_2$

$\therefore$ The complete solution is

$$y = vx = x^2 e^x - xe^x + c_1 x\, e^x + c_2 x.$$

Example 34. *Solve:* $\sin^2 x \,.\, \frac{d^2y}{dx^2} = 2y$ *given* $y = \cot x$ *is a solution.*

Sol. Putting $y = v \cot x$, so that

$$\frac{dy}{dx} = \frac{dv}{dx}\cot x - v\,\text{cosec}^2\, x$$

and

$$\frac{d^2y}{dx^2} = \frac{d^2v}{dx^2}\cot x - 2\,\text{cosec}^2\, x\,\frac{dv}{dx} + 2v\,\text{cosec}^2\, x\cot x$$

in the given equation, we get

$$\cot x \sin^2 x \frac{d^2v}{dx^2} - 2\frac{dv}{dx} = 0$$

or $$\frac{d^2y}{dx^2} - \frac{2}{\sin x \cos x}\frac{dv}{dx} = 0$$

or $$\frac{dp}{dx} = \frac{2}{\sin x \cos x}\, p, \quad \text{where } p = \frac{dv}{dx}$$

or $$\frac{dp}{p} = \frac{2}{\sin x \cos x}\, dx = \frac{2\sec^2 x}{\tan x}\, dx$$

Integrating, we get

$$\log p = 2 \log \tan x + \log c \qquad \therefore \qquad p = c_1 \tan^2 x$$

or $$\frac{dv}{dx} = c_1 \tan^2 x = c_1(\sec^2 x - 1)$$

Integrating, $\quad v = c_1 (\tan x - x) + c_2$

∴ The complete solution is

$$y = v \cot x = c_1(1 - x \cot x) + c_2 \cot x.$$

Example 35. *Solve :* $x\dfrac{dy}{dx} - y = (x-1)\left(\dfrac{d^2y}{dx^2} - x + 1\right).$

Sol. The given equation may be written as

$$\frac{d^2y}{dx^2} - \frac{x}{x-1}\cdot\frac{dy}{dx} + \frac{y}{x-1} = x - 1$$

Here $\;P + Qx = 0$

∴ $\quad y = x$ is a part of C.F.

∴ Putting $y = vx$, so that $\dfrac{dy}{dx} = \dfrac{dv}{dx}x + v \quad$ and $\quad \dfrac{d^2y}{dx^2} = \dfrac{d^2v}{dx^2}x + 2\dfrac{dv}{dx}$

We have, $$\frac{d^2v}{dx^2} + \left(-\frac{x}{x-1} + \frac{2}{x}\right)\frac{dv}{dx} = \frac{x-1}{x}$$

or $$\frac{dp}{dx} + \left(-\frac{x}{x-1} + \frac{2}{x}\right)p = \frac{x-1}{x}, \quad \text{where } p = \frac{dv}{dx}$$

which is a linear equation.

$$\text{I.F.} = e^{-\int \frac{x}{x-1}dx + \int \frac{2}{x}dx} = e^{-\int\left(1+\frac{1}{x-1}\right)dx + \int \frac{2}{x}dx}$$

$$= e^{-x - \log(x-1) + 2\log x} = \frac{x^2}{x-1}e^x$$

$$\therefore \quad p\,\frac{x^2e^{-x}}{x-1} = \int \frac{x-1}{x}\cdot\frac{x^2}{x-1}e^{-x}\,dx + c_1 = \int xe^{-x}\,dx + c_1 = -xe^{-x} - e^{-x} + c_1$$

$$\therefore \quad p = \frac{dv}{dx} = -\frac{x-1}{x} - \frac{(x-1)}{x^2} + \frac{c_1(x-1)e^x}{x^2} = -1 + \frac{1}{x^2} + c_1\left(\frac{1}{x} - \frac{1}{x^2}\right)e^x$$

NOTES

Integrating, $v = -x - \frac{1}{x} + c_1 \frac{1}{x} e^x + c_2$

$\therefore$ The complete solution is

$$y = vx = -x^2 - 1 + c_1e^x + c_2x = c_1e^x + c_2x - (1 + x^2).$$

Example 36. *Solve:*

$$(x \sin x + \cos x) \frac{d^2y}{dx^2} - x \cos x \frac{dy}{dx} + y \cos x = \sin x\, (x \sin x + \cos x)^2.$$

Sol. The given equation may be written as

$$\frac{d^2y}{dx^2} - \frac{x \cos x}{x \sin x + \cos x} \cdot \frac{dy}{dx} + \frac{\cos x}{x \sin x + \cos x}\, y = \sin x(x \sin x + \cos x)$$

Here $P + Qx = 0$ $\therefore$ $y = x$ is a part of C.F.

$\therefore$ Putting $y = vx$ the equation reduces to

$$\frac{d^2v}{dx^2} + \left(\frac{2}{x} - \frac{x \cos x}{x \sin x + \cos x}\right) \frac{dv}{dx} = \frac{\sin x\, (x \sin x + \cos x)}{x}$$

or
$$\frac{dp}{dx} + \left(\frac{2}{x} - \frac{x \cos x}{x \sin x + \cos x}\right) p = \frac{\sin x}{x}\, (x \sin x + \cos x)$$

which is a linear equation.

$$\therefore \quad \text{I.F.} = e^{\int \left(\frac{2}{x} - \frac{x \cos x}{x \sin x + \cos x}\right) dx} = e^{2 \log x - \log (x \sin x + \cos x)} = \frac{x^2}{(x \sin x + \cos x)}$$

$$\therefore \quad p.\, \frac{x^2}{(x \sin x + \cos x)} = \int x \sin x\, dx + c_1 = -x \cos x + \sin x + c_1$$

$$\therefore \quad p = \frac{dv}{dx} = \frac{1}{x^2} (-x \cos x + \sin x)(x \sin x + \cos x) + \frac{c_1}{x^2} (x \sin x + \cos x)$$

$$\frac{dy}{dx} = -\sin x \cos x - \frac{1}{x} \cos 2x + \frac{1}{x^2} \sin x \cos x + c_1 \left(\frac{1}{x} \sin x + \frac{1}{x^2} \cos x\right)$$

Integrating,

$$v = \frac{1}{2} \cos^2 x - \int \frac{1}{x} \cos 2x\, dx + \int \frac{1}{2x^2} \sin 2x\, dx + c_1 \int \left(\frac{1}{x} \sin x + \frac{1}{x^2} \cos x\right) dx$$

$$= \frac{1}{2} \cos^2 x - \frac{1}{2x} \sin 2x - \frac{c_1}{x} \cos x + c_2$$

$\therefore$ The complete solution is

$$y = vx = \frac{x}{2} \cos^2 x - \frac{1}{2} \sin 2x - c_1 \cos x + c_2\, x.$$

Example 37. *Solve:* $(1 - x^2) \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y = x(1 - x^2)^{3/2}$.

Sol. The given equation can be written as

$$\frac{d^2y}{dx^2} + \frac{x}{1 - x^2} \frac{dv}{dx} - \frac{1}{1 - x^2}\, y = x(1 - x^2)^{1/2}$$

Here P + Qx = 0. $\therefore$ $y = x$ is a part of C.F.

Putting $y = vx$, the equation reduces to

$$\frac{d^2v}{dx^2} + \left(\frac{x}{1-x^2} + \frac{2}{x}\right)\frac{dv}{dx} = \frac{x(1-x^2)^{1/2}}{x}$$

or
$$\frac{dp}{dx} + \left(\frac{x}{1-x^2} + \frac{2}{x}\right)p = \sqrt{1-x^2}\text{, where } p = \frac{dv}{dx}$$

which is a linear equation.

$$\text{I.F.} = e^{\int\left(\frac{x}{1-x^2}+\frac{2}{x}\right)dx} = e^{-\frac{1}{2}\log(1-x^2)+2\log x} = \frac{x^2}{\sqrt{1-x^2}}$$

$$\therefore \quad p.\frac{x}{\sqrt{1-x^2}} = \int x^2\,dx + c_1 = \frac{x^3}{3} + c_1$$

$$\therefore \quad p = \frac{dv}{dx} = \frac{1}{3}x\sqrt{1-x^2} + \frac{c_1}{x^2}\sqrt{1-x^2}$$

$$= \frac{1}{3}x\sqrt{1-x^2} + c_1(1-x^2)^{1/2}\cdot\frac{1}{x^2}$$

Integrating,
$$v = -\frac{1}{9}(1-x^2)^{3/2} + c_1(1-x^2)^{1/2}\left(-\frac{1}{x}\right) - c_1\int\frac{dx}{\sqrt{1-x^2}} + c_2$$

$$= -\frac{1}{9}(1-x^2)^{3/2} - \frac{c_1}{x}(1-x^2)^{1/2} - c_1\sin^{-1}x + c_2$$

$\therefore$ The complete solution is

$$y = vx = -\frac{1}{9}x(1-x^2)^{1/2} - c_1\{x\sin^{-1}x + \sqrt{1-x^2}\} + c_2x.$$

Example 38. *Solve* $\frac{d^2y}{dx^2} - \cot x\,\frac{dy}{dx} - (1-\cot x)\,y = e^x \sin x.$

Sol. From the above equation, we have P + Q + 1 = 0

$\therefore$ $y = e^x$ is a part of C.F.

$\therefore$ Putting $y = ve^x$ so that $\frac{dy}{dx} = \frac{dv}{dx}e^x + v.\,e^x$

and
$$\frac{d^2y}{dx^2} = \frac{d^2v}{dx^2}e^x + 2\frac{dv}{dx}e^x + ve^x$$

We have,
$$\frac{d^2v}{dx^2} + (2-\cot x)\frac{dv}{dx} = \sin x$$

or
$$\frac{dp}{dx} + (2-\cot x)\,p = \sin x\text{, where } p = \frac{dv}{dx}$$

which is linear equation.

$$\text{I.F.} = e^{\int(2-\cot x)\,dx} = e^{2x-\log\sin x} = \frac{e^{2x}}{\sin x}$$

$$\therefore \quad p\,\frac{e^{2x}}{\sin x} = \int\frac{e^{2x}}{\sin x}\sin x\,dx + c_1 = \frac{1}{2}e^{2x} + c_1$$

NOTES

$\therefore \qquad p\ \dfrac{dv}{dx} = \dfrac{1}{2} \sin x + c_1 e^{-2x} \sin x$

Integrating, $\qquad v = -\dfrac{1}{2} \cos x + \dfrac{c_1}{5} e^{-2x} (-2 \sin x - \cos x) + c_2$

$\therefore$ The complete solution is

$$y = ve^x = -\frac{1}{2} e^x \cos x - \frac{c_1}{5} e^{-x} (2 \sin x + \cos x) + c_2 e^x.$$

Example 39. *Solve:* $(x + 2) \dfrac{d^2y}{dx^2} - (2x + 5) \dfrac{dy}{dx} + 2y = (x + 1) e^x.$

Sol. The above given equation may be written as

$$\frac{d^2y}{dx^2} - \frac{2x+5}{x+2} \cdot \frac{dy}{dx} + \frac{2}{x+2} \cdot y = \frac{x+1}{x+2} e^x$$

Here, $\qquad \dfrac{P}{2} + \dfrac{Q}{2^2} + 1 = 0$

$\therefore$ $y = e^{2x}$ is a solution of this equation.

$\therefore$ Putting $y = ve^{2x}$, the equation reduces to

$$(x + 2) \frac{d^2v}{dx^2} + (2x + 3) \frac{dv}{dx} = (x + 1) e^{-x}$$

$$\frac{d^2v}{dx^2} + \frac{2x+3}{x+2} \frac{dv}{dx} = \frac{x+1}{x+2} e^{-x}$$

or $\qquad \dfrac{dp}{dx} + \dfrac{2x+3}{x+2} p = \dfrac{x+1}{x+2} e^{-x}, \quad$ where $p = \dfrac{dv}{dx}$

which is a linear equation.

$$\text{I.F.} = e^{\int \frac{2x+3}{x+2} dx} = e^{\int \left(2 - \frac{1}{x+2}\right) dx} = e^{2x - \log (x+2)} = \frac{e^{2x}}{x+2}$$

$\therefore \qquad p \cdot \dfrac{e^{2x}}{x+2} = \displaystyle\int \frac{x+1}{(x+2)^2} e^x\, dx + c_1$

$$= \int \left\{ \frac{1}{x+2} - \frac{1}{(x+2)^2} \right\} e^x\, dx + c_1 = \frac{e^x}{x+2} + c_1$$

$\therefore \qquad p = \dfrac{dv}{dx} = e^{-x} + c_1\, e^{-2x} (x + 2).$

Integrating, $\qquad v = -e^{-x} - \tfrac{1}{2} c_1 e^{-2x}(x + 2) - \tfrac{1}{4} c_1 e^{-2x} + c_2$

$$= -e^{-x} - \tfrac{1}{4} c_1 (2x + 5) e^{-2x} + c_2$$

$\therefore$ The complete solution is

$$y = ve^{2x} = -e^x - \tfrac{1}{4} c_1 (2x + 5) + c_2 e^{2x}.$$

Example 40. *Solve :* $x \dfrac{d^2y}{dx^2} - (2x + 1) \dfrac{dy}{dx} + (x + 1)y = (x^2 + x - 1)e^{2x}.$

Sol. The given equation can be written as

$$\frac{d^2y}{dx^2} - \left(2 + \frac{1}{x}\right) \frac{dy}{dx} + \left(1 + \frac{1}{x}\right) y = \left(x + 1 - \frac{1}{x}\right) e^{2x}$$

Here $1 + P + Q = 0 \therefore y = e^x$ is a part of C.F.

Putting $y = ve^x$ the equation reduces to

$$\frac{d^2v}{dx^2} - \frac{1}{x}\frac{dv}{dx} = \left(x + 1 - \frac{1}{x}\right) e^{xs}$$

or $$\frac{dp}{dx} - \frac{1}{x} p = \left(x + 1 - \frac{1}{x}\right) e^x, \quad \text{where } p = \frac{dv}{dx}$$

which is a linear equation.

$$\text{I.F.} = e^{-\int \frac{1}{x} dx} = e^{-\log x} = \frac{1}{x}$$

$$\therefore \quad p.\frac{1}{x} = \int \left(x + 1 - \frac{1}{x}\right) e^x . \frac{1}{x} dx + k$$

$$= \int \left(e^x + \frac{1}{x} e^x - \frac{1}{x^2} e^x\right) dx + k = e^x + \frac{e^x}{x} + k$$

$$\therefore \quad p = \frac{dv}{dx} = xe^x + e^x + kx,$$

Integrating, $v = xe^x + \frac{k}{2} x^2 + c_2$ or $v = xe^x + c_1x^2 + c_2$, where $c_1 = \frac{k}{2}$

$\therefore$ The complete solution is

$$y = ve^x = xe^{2x} + c_1x^2e^x + c_2e^x.$$

EXERCISE E

Solve the following differential equations:

1. $x\frac{d^2y}{dx^2} - (3 + x)\frac{dy}{dx} + 3y = 0$
2. $x\frac{d^2y}{dx^2} - (2x - 1)\frac{dy}{dx} + (x - 1)y = 0$
3. $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$, given that $y = x + \frac{1}{x}$ is a solution.
4. $(2 + x)y'' - (9 + 4x)y' + (7 + 3x)y = 0$
5. $x(x \cos x - 2 \sin x)y'' + (x^2 + 2)\sin x . y' - 2(x \sin x + \cos x)y = 0$
6. $x\frac{d^2y}{dx^2} - (x + 2)\frac{dy}{dx} + 2y = x^3.$
7. $x\frac{d^2y}{dx^2} - 2(x + 1)\frac{dy}{dx} + (x + 2)y = (x - 2)e^x.$
8. $\frac{d^2y}{dx^2} - (1 + x)\frac{dy}{dx} + xy = x$
9. $(x + 1)\frac{d^2y}{dx^2} - 2(x + 3)\frac{dy}{dx} + (x + 5) = e^x$
10. $(x - x^2)y'' - (1 - 2x)y' + (1 - 3x + x^2)y = (1 - x)^3$

Answers

1. $y = - c_1(x^3 + 3x^2 + 6x + 6) + c_2e^x$
2. $y = (c_1 \log x + c_2)e^x$
3. $y = \frac{c_1'}{x} + c_2\left(x + \frac{1}{x}\right)$
4. $y = c_1(2x + 3)e^{3x} + c_2e^x$
5. $y = c_1 \sin x + c_2x^2$
6. $y = c_1(x^2 + 2x + 2) + c_2e^x - x^3$
7. $y = -\frac{1}{2}x^2e^x + xe^x + \frac{1}{3}c_1x^3e^x + c_2e^x$
8. $y = c_1e^x \int e^{-x+\frac{1}{2}x^2} dx + c_2e^x + 1$
9. $y = -\frac{1}{4}xe^x + \frac{1}{5}c_1e^x(x + 1)^5 + c_2e^x$
10. $y = \frac{1}{2}c_1x^2e^{-x} + c_2e^x - x$

REMOVAL OF THE FIRST DERIVATIVE (Ruduction to Normal Form)

NOTES

If the part of the complementary function is not obvious by inspection, it is sometimes useful to reduce the given equation into the form in which the term containing the first derivative is absent. For this we will change the **dependent variable** in the equation.

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} = Qy + R \qquad ...(1)$$

By putting $y = uv$, where u is some function of x, so that

$$\frac{dy}{dx} = u\frac{dv}{dx} + \frac{du}{dx}v$$

and
$$\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\cdot\frac{dv}{dx} + \frac{d^2u}{dx^2}v$$

∴ Equation (1) reduces to

$$u\frac{d^2v}{dx^2} + \left(Pu + 2\frac{du}{dx}\right)\frac{dv}{dx} + \left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right)v = R$$

or
$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} + \left(\frac{1}{u}\frac{d^2u}{dx^2} + \frac{P}{u}\frac{du}{dx} + Q\right)v = \frac{R}{u} \qquad ...(2)$$

Let us choose u such that $\quad P + \frac{2}{u}\frac{du}{dx} = 0$

or
$$\frac{du}{dx} = -\frac{P}{2}u \quad \text{or} \quad \frac{du}{u} = -\frac{1}{2}Pdx$$

∴
$$\boldsymbol{u = e^{-\frac{1}{2}\int P dx}}$$

∴ From equation (2), we have

$$\frac{d^2v}{dx^2} + \left[\frac{1}{u}\left(-\frac{u}{2}\frac{dP}{dx} - \frac{P}{2}\frac{du}{dx}\right) + \frac{P}{u}\frac{du}{dx} + Q\right]v = Re^{\frac{1}{2}\int P\,dx}$$

or
$$\frac{d^2v}{dx^2} + \left[-\frac{1}{2}\frac{dP}{dx} - \frac{P}{2u}\left(-\frac{P}{2}u\right) + \frac{P}{u}\left(-\frac{P}{2}u\right) + Q\right]v = Re^{\frac{1}{2}\int P\,dx}$$

or
$$\frac{d^2v}{dx^2} + \left[Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2\right]v = R.e^{\frac{1}{2}\int P\,dx}$$

or
$$\boldsymbol{\frac{d^2v}{dx^2} + Xv = Y} \qquad ...(3)$$

where $\boldsymbol{X = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2}$ and $\boldsymbol{Y = Re^{-\frac{1}{2}\int P\,dx}}$

The equation (3), may easily be integrated. Equation (3) is said to be the **normal form** of the equation (1).

Note. Remember equation (3), and the values of *u*, X and Y.

SOLVED EXAMPLES

Example 41. *Solve:* $\frac{d^2y}{dx^2} + \frac{1}{x^{1/3}}\frac{dy}{dx} + \left(\frac{1}{4x^{2/3}} - \frac{1}{6x^{4/3}} - \frac{6}{x^2}\right) y = 0.$

Sol. Here $P = x^{-1/3}$, $Q = \frac{1}{4x^{2/3}} - \frac{1}{6x^{4/3}} - \frac{6}{x^2}$ and $R = 0$

On putting $y = uv$, the given equation reduces to the normal form

$$\frac{d^2v}{dx^2} + Xv = Y$$

where $u = e^{-\frac{1}{2}\int P\,dx} = e^{-\frac{1}{2}\int x^{-1/3}\,dx} = e^{-\frac{1}{3}x^{2/3}}$

$$X = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2$$

$$= \frac{1}{4x^{2/3}} - \frac{1}{6x^{4/3}} - \frac{6}{x^2} - \frac{1}{2}\left(-\frac{1}{3}x^{-4/3}\right) - \frac{1}{4}x^{-2/3} = -\frac{6}{x^2}$$

and $Y = Re^{\frac{1}{2}\int P\,dx} = 0$

$\therefore$ The normal form of the given equation is

$$\frac{d^2v}{dx^2} - \frac{6}{x^2}v = 0 \quad \text{or} \quad x^2\frac{d^2v}{dx^2} - 6v = 0 \qquad \ldots(1)$$

which is a homogeneous linear equation.

Putting $x = e^z$, so that $\frac{dx}{dz} = e^z = x$

$\therefore$ $\frac{dv}{dz} = \frac{dv}{dx}\cdot\frac{dx}{dz} = x\frac{dv}{dx}$

$\therefore$ $x\frac{d}{dx} \equiv \frac{d}{dz}$

Let D stands for $\frac{d}{dz}$, then

$$x\frac{d}{dx}\left(x\frac{dv}{dx}\right) = x^2\frac{d^2v}{dx^2} + x\frac{dv}{dx}$$

or $$x^2\frac{d^2v}{dx^2} = \left(x\frac{d}{dx} - 1\right)x\frac{dv}{dx} = (D-1)\,Dv$$

$\therefore$ From equation (1), we get $[(D-1)\,D - 6]\,v = 0$

or $(D^2 - D - 6)\,v = 0$ ...(2)

Now, A.E. is $m^2 - m - 6 = 0$ or $m = 3, -2$

Solution of (2) is $v = c_1e^{3z} + c_2e^{-2z} = c_1x^3 + c_2x^{-2}$

$\therefore$ The solution of the given equation is

$$y = uv = e^{\left(-\frac{3}{4}\right)x^{2/3}}(c_1x^3 + c_2x^{-2}).$$

Example 42. *Solve:* $x^2\frac{d^2y}{dx^2} - 2(x^2 + x)\frac{dy}{dx} + (x^2 + 2x + 2)y = 0.$

Sol. The given equation can be written as

$$\frac{d^2y}{dx^2} - 2\left(1 + \frac{1}{x}\right)\frac{dy}{dx} + \left(1 + \frac{2}{x} + \frac{2}{x^2}\right)y = 0$$

NOTES

Here $P = -2\left(1+\frac{1}{x}\right)$, $Q = 1 + \frac{2}{x} + \frac{2}{x^2}$ and $R = 0$

Putting $y = uv$, the normal form is $\frac{d^2v}{dx^2} + Xv + Y$

where $u = e^{-\frac{1}{2}\int P dx} = e^{\int\left(1+\frac{1}{x}\right)dx} = e^{x + \log x} = xe^x$

$$X = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2 = 1 + \frac{2}{x} + \frac{2}{x^2} - \frac{1}{2}.\frac{2}{x^2} - \frac{1}{4}4\left(1+\frac{1}{x}\right)^2 = 0$$

and $Y = Re^{\frac{1}{2}\int P dx} = 0$

$\therefore$ The normal form is $\frac{d^2v}{dx^2} = 0$

Integrating, $v = c_1x + c_2$

$\therefore$ The solution of the given equation is $y = uv = xe^x (c_1x + c_2)$.

Example 43. *Solve* $\frac{d^2y}{dx^2} - 2\tan x\frac{dy}{dx} + 5y = e^x \sec x$.

Sol. Here $P = -2\tan x$, $Q = 5$ and $R = e^x \sec x$

Putting $y = uv$ is the given equation, the equation reduces to

$$\frac{d^2v}{dx^2} + Xv = Y, \text{ where } u = e^{-\frac{1}{2}\int P dx}$$

$$= e^{\int \tan x\, dx} = e^{\log \sec x} = \sec x$$

$$X = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2$$

$$= 5 + \frac{1}{2}2\sec^2 x - \frac{1}{4}.4\tan^2 x = 6$$

$$Y = Re^{\frac{1}{2}\int P dx} = e^x$$

$\therefore$ The reduced equation is $\frac{d^2v}{dx^2} + 6v = e^x$

where C.F. $= c_1 \cos \sqrt{6}x + c_2 \sin \sqrt{6}x$

and $\text{P.I.} = \frac{1}{D^2+6}e^x = \frac{e^x}{7}$

$\therefore$ $v = c_1 \cos \sqrt{6}x + c_2 \sin \sqrt{6}x + \frac{e^x}{7}$

$\therefore$ The solution of the given equation is

$$y = uv = \sec x\left(c_1 \cos \sqrt{6}x + c_2 \sin \sqrt{6}x + \frac{1}{7}e^x\right).$$

Example 44. *Solve* $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 3)y = e^{x^2}$.

Sol. Here $P = -4x$, $Q = 4x^2 - 3$, $R = e^{x^2}$

Putting $y = uv$, the normal form is,

$$\frac{d^2v}{dx^2} + Xv = Y, \text{ where } u = e^{-\frac{1}{2}\int P dx} = e^{x^2}$$

$$X = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P_2 = 4x^2 - 3 - \frac{1}{2}(-4) - \frac{1}{4}(16x^2) = -1$$

and $Y = Re^{-\frac{1}{2}\int P dx} = 1$

$\therefore$ The normal form is $\dfrac{d^2v}{dx^2} - v = 1$

where C.F. $= c_1e^x + c_2e^{-x}$

and P.I. $= \dfrac{1}{D^2 - 1} \cdot 1 = -(1 - D^2)^{-1} \cdot 1 = -1$

$\therefore$ $$v = c_1e^x + c_2e^{-x} - 1$$

$\therefore$ The solution of the given equation is

$$y = uv = e^{x^2} (c_1e^x + c_2e^{-x} - 1).$$

Example 45. *Solve* $\dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + (4x^2 - 1)\,y = -3e^{x^2}\sin 2x.$

Sol. Here $P = -4x$, $Q = 4x^2 - 1$ and $R = -3e^{x^2}\sin 2x$

Putting $y = uv$, the equation reduces to $\dfrac{d^2v}{dx^2} + Xv = Y$

where $u = e^{-\frac{1}{2}\int P dx} = e^{x^2}$

$$X = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2 = 4x^2 - 1 - \frac{1}{2}(-4) - \frac{1}{4}16x^2 = 1$$

$$Y = Re^{\frac{1}{2}\int P dx} = -3\sin 2x.$$

$\therefore$ The reduced equation is $\dfrac{d^2v}{dx^2} + v = -3\sin 2x$

whose C.F. $= c_1\cos x + c_2\sin x$

and P.I. $= \dfrac{1}{D^2 + 1}(-3\sin 2x) = \dfrac{-3}{-2^2 + 1}\sin 2x = \sin 2x$

$\therefore$ $$v = c_1\cos x + c_2\sin x + \sin 2x$$

$\therefore$ The solution of the given equation is

$$y = uv = e^{x^2}(c_1\cos x + c_2\sin x + \sin 2x).$$

Example 46. *Solve* $\dfrac{d^2y}{dx^2} - \dfrac{2}{x}\dfrac{dy}{dx} + \left(1 + \dfrac{2}{x^2}\right)y = xe^x.$

Sol. Here $P = -\dfrac{2}{x}$, $Q = 1 + \dfrac{2}{x^2}$ and $R = xe^x$

Putting $y = uv$, the normal form is $\dfrac{d^2v}{dx^2} + Xv = Y$

where $u = e^{-\frac{1}{2}\int P dx} = e^{\int 1/x\, dx} = e^{\log x} = x$

$$X = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2 = 1 \text{ and } Y = Re^{\frac{1}{2}\int P dx}$$

$$= xe^x e^{-\int dx/x} = xe^x e^{-\log x} = e^x$$

$\therefore$ The normal form of the given equation is

$$\frac{d^2v}{dx^2} + v = e^x \text{ whose C.F.} = c_1\cos x + c_2\sin x$$

and $$\text{P.I.} = \frac{1}{D^2 + 1}e^x = \frac{e^x}{2}$$

NOTES

$\therefore \qquad v = c_1 \cos x + c_2 \sin x + \frac{1}{2} e^x$

$\therefore$ The solution of the given equation is

$$y = uv = x(c_1 \cos x + c_2 \sin x + \tfrac{1}{2} e^x).$$

Example 47. *Solve*

$$x^2 (\log x)^2 \frac{d^2y}{dx^2} - 2x \log x \frac{dy}{dx} + [2 + \log x - 2(\log x)^2]\, y = (\log x)^3 x^2.$$

Sol. The given equation can be written as

$$\frac{d^2y}{dx^2} - \frac{2}{x \log x}\frac{dy}{dx} + \left[\frac{2}{x^2 (\log x)^2} + \frac{1}{x^2 \log x} - \frac{2}{x^2}\right] y = \log x$$

Here, $\qquad P = -\frac{2}{x \log x}$, $Q = \frac{2}{x^2 (\log x)^2} + \frac{1}{x^2 \log x} - \frac{2}{x^2}$

and $\qquad R = \log x$

Putting $y = uv$, the given equation is transformed to

$$\frac{d^2v}{dx^2} + Xv = Y$$

where $u = e^{-\frac{1}{2}\int P\,dx} = e^{\int \frac{1}{x \log x} dx} = e^{(\log \log x)} = \log x$

$$X = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2$$

$$= \frac{2}{x^2 (\log x)^2} + \frac{1}{x^2 \log x} - \frac{2}{x^2} - \frac{1}{2}\frac{2(\log x + 1)}{x^2 (\log x)^2} - \frac{1}{4}\frac{4}{(x \log x)^2} = \frac{2}{x^2}$$

and $\qquad Y = Re^{\frac{1}{2}\int P dx} = 1$

$\therefore$ The transformed equation is

$$\frac{d^2v}{dx^2} - \frac{2}{x^2} v = 1$$

$$x^2 \frac{d^2v}{dx^2} - 2v = x^2 \qquad \ldots(1)$$

Putting $x = e^z$, we get $\{D(D-1) - 2\}\, v = e^{2z}$ or $(D^2 - D - 2)\, v = e^{2z}$

A.E. is $m^2 - m - 2 = 0$, $m = 2, -1$

C.F. $= c_1 e^{2z} + c_2 e^{-z} = c_1 x^2 + c_2 x^{-1}$

and $\qquad \text{P.I.} = \frac{1}{D^2 - D - 2} e^{2z} = \frac{1}{(D-2)(D+1)} e^{2z}$

$$= \frac{1}{3}\frac{1}{D-2}(e^{2z} \,.\, 1) = \frac{1}{3} e^{2z} \frac{1}{D+2-2} \,.\, 1$$

$$= \frac{1}{3} e^{2z}\left(\frac{1}{D} \,.\, 1\right) = \frac{1}{3} z e^{2z} = \frac{1}{3} x^2 \log x$$

$\therefore$ The solution of equation (1) is

$$v = c_1 x^2 + c_2 x^{-1} + \frac{1}{3} x^2 \log x$$

∴ The solution of the given equation is

$$y = uv = (\log x)\,(c_1 x^2 + c_2 x^{-1}) + \frac{1}{3}(x \log x)^2.$$

Example 48. *Solve* $\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} + (x^2 + 1)y = x^3 + 3x.$

Sol. Here $P = 2x$, $Q = x^2 + 1$ and $R = x^3 + 3x$

Putting $y = uv$, the equation is transformed to $\dfrac{d^2v}{dx^2} + Xv = y,$

where $u = e^{-\frac{1}{2}\int P\,dx}\; e^{-x^2/2}$

$$X = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2 = 0$$

and $Y = Re^{-\frac{1}{2}\int P\,dx} = (x^3 + 3x)\,e^{x^{2/3}}$

∴ The transformed equation is, $\dfrac{d^2v}{dx^2} = (x^3 + 3x)\;e^{x^{2/3}}$

Integrating, $\dfrac{dv}{dx} = \displaystyle\int x^3\, e^{x^2/2}\,dx + 3\int xe^{x^2/2}\,dx + c_1$

$$= \int x^2\,(xe^{x^2/2})\,dx + 3e^{x^2/2} + c_1$$

$$= x^2e^{-x^2/2} - 2\int xe^{x^2/2}\,dx + 3e^{x^2/2} + c_1$$

$$= x^2e^{x^2/2} - 2e^{x^2/2} + 3e^{x^2/2} + c_1 = (x^2 + 1)e^{x^2/2} + c_1$$

Integrating again $= \displaystyle\int x^2\, e^{x^2/2}\,dx + \int e^{x^2/2}\,dx + c_1\,x + c_2$

$$= \int x\,(xe^{x^2/2})\,dx + \int e^{x^2/2}\,dx + c_1x + c_2 = xe^{x^2/2} + c_1x + c_2$$

∴ The solution of the given equation is

$$y = uv = x + (c_1x + c_2)e^{x^2/2}$$

Example 49. *Solve* $\dfrac{d^2y}{dx^2} - \dfrac{1}{\sqrt{x}}\dfrac{dy}{dx} + \dfrac{1}{4x^2}(-8 + \sqrt{x} + x)\,y = 0.$

Sol. Here, $P = -\dfrac{1}{\sqrt{x}},\; Q = \dfrac{1}{4x^2}(-8 + \sqrt{x} + x)$ and $R = 0$

Putting $y = uv$, the given equation is transformed to

$$\frac{d^2v}{dx^2} + Xv = Y$$

where $u = e^{-\frac{1}{2}\int P\,dx} = e^{-\frac{1}{2}\int \frac{1}{\sqrt{x}}dx} = e^{\sqrt{x}}$

$$X = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2 = -\frac{2}{x^2}$$

$$Y = Re^{\frac{1}{2}\int P\,dx} = 0$$

NOTES

∴ The transformed equation is

$$\frac{d^2v}{dx^2} - \frac{2}{x^2}v = 0 \qquad \text{or} \qquad x^2\frac{d^2v}{dx^2} - 2v = 0$$

which is a homogeneous linear equation.

$$\{D(D-1) - 2\}\, v = 0$$

$$(D^2 - D - 2)\, v = 0$$

A.E. is $\qquad m^2 - m - 2 = 0$

$$m = 2, -1$$

∴ $\qquad v = c_1 e^{2z} + c_2 e^{-z} = c_1 x^2 + c_2 x^{-1}.$

∴ The solution of the given equation is

$$y = uv = e^{\sqrt{x}} \,.\, (c_1 x^2 + c_2 x^{-1}).$$

Example 50. *Solve* $\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} + (x^2 + 5)\, y = xe^{-\frac{1}{2}x^2}$.

Sol. Here $P = 2x$, $\quad Q = x^2 + 5$, $\quad R = xe^{-\frac{1}{2}x^2}$.

Putting $y = uv$, the given equation is transformed to

$$\frac{d^2v}{dx^2} + Xv = Y$$

where $\qquad u = e^{-\frac{1}{2}\int P dx} = e^{-\frac{1}{2}\int 2x\, dx} - e^{-x^2/2}$

$$X = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2 = x^2 + 5 - 1 - x^2 = 4$$

and $\qquad Y = Re^{\frac{1}{2}\int P dx} = x.$

∴ The transformed equation is

$$\frac{d^2v}{dx^2} + 4v = x$$

A.E. is $\qquad m^2 + 4 = 0 \quad \therefore \quad m = \pm 2i$

$$\text{C.F.} = c_1 \cos(2x + c_2)$$

$$\text{P.I.} = \frac{1}{D^2 + 4}x = \frac{1}{4}\left(1 + \frac{D^2}{4}\right)^{-1} x = \frac{x}{4}$$

∴ The solution is $\qquad y = uv = e^{-x^2/2}\, [c_1 \cos(2x + c_2) + \tfrac{1}{4}x].$

EXERCISE F

Solve the following differential equations:

1. $\dfrac{d^2y}{dx^2} - \dfrac{2}{x}\left(\dfrac{dy}{dx}\right) + \left(a^2 + \dfrac{2}{x^2}\right)y = 0$
2. $(x^3 - 2x^2)\dfrac{d^2y}{dx^2} + 2x^2\dfrac{dy}{dx} - 12(x-2)\, y = 0$
3. $\dfrac{d^2y}{dx^2} - 2\tan x\dfrac{dy}{dx} + 5y = 0$
4. $\dfrac{d^2y}{dx^2} + \dfrac{2}{x}\dfrac{dy}{dx} + n^2y = 0$
5. $\dfrac{d^2y}{dx^2} - 2bx\dfrac{dy}{dx} + b^2x^2y = 0$
6. $x\dfrac{d}{dx}\left(x\dfrac{dy}{dx} - y\right) - 2x\dfrac{dy}{dx} + 2y + x^2y = 0$
7. $\dfrac{d}{dx}\left(\cos^2 x\dfrac{dy}{dx}\right) + y\cos^2 x = 0$
8. $\left(\dfrac{d^2y}{dx^2} + y\right)\cot x + 2\left(\dfrac{dy}{dx} + y\tan x\right) = \sec x$

9. $\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (x^2 + 2)y = e^{\frac{1}{2}(x^2+2x)}$ 10. $\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + (x^2 - 8)y = x^2 e^{-\frac{x^2}{2}}$

11. $x^2y'' - 2x(1 + x)y' + 2(1 + x)y = x^3\ (x > 0)$ 12. $y'' - (2\cot x)y' + (1 + 2\cot^2 x)y = 0$

13. $y'' - 4xy + (4x^2 - 1)y = e^{x^2}(5 - 3\cos 2x)$

Answers

1. $y = xc_1 \cos(ax + c_2)$ 2. $y = (c_1x^4 + c_2 x^{-3})/(x - 2)$

3. $y = \sec x(c_1 \cos\sqrt{6}x + c_2 \sin\sqrt{6}x)$ 4. $y = \frac{1}{x} c_1 \cos(nx + c_2)$

5. $y = c_1 e^{\frac{1}{2}bx^2} \cos(\sqrt{6}x + c_2)$ 6. $y = x(c_1 \cos x + c_2 \sin x)$

7. $y = \sec x\left(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x\right)$ 8. $y = \frac{1}{2}(\sin x - x\cos x) + (c_1x + c_2)\cos x$

9. $y = e^{\frac{x^2}{z}}\left(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x\right) + \frac{1}{4}e^{\frac{1}{2}(x^2+2x)}$ 10. $y = e^{-\frac{x^2}{2}}\left[c_1e^{3x} + c_2e^{-3x} - \frac{1}{9}\left(x^2 + \frac{2}{9}\right)\right]$

11. $y = (c_1e^{2x} + c_2)x - \frac{x^2}{2}$ 12. $y = (c_1 + c_2x)\sin x$

13. $y = e^{x^2}(c_1 \cos x + c_2 \sin x + 5 + \cos 2x)$

TRANSFORMATION OF THE EQUATION BY CHANGING THE INDEPENDENT VARIABLE

Sometimes the equation is transformed to an integrable form by **changing the independent variable.**

Let the equation be

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad ...(1)$$

Let the independent variable be changed from x to z, where z is a function of x.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dz}\cdot\frac{dz}{dx}\right)$$

$$= \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dx}\cdot\frac{d^2z}{dx^2}$$

Substituting in equation (1), we have

$$\left(\frac{dz}{dx}\right)^2\frac{d^2y}{dz^2} + \left(\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right)\frac{dy}{dz} + Qy = R$$

or

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1y = R_1 \qquad ...(2)$$

where $P_1 = \dfrac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2}$, $Q_1 = \dfrac{Q}{\left(\dfrac{dz}{dx}\right)^2}$ and $R_1 = \dfrac{R}{\left(\dfrac{dz}{dx}\right)^2}$

P_1, Q_1, R_1 are functions of x but may be expressed as functions of z by the given relation between z and x.

NOTES

We choose z to make the co-efficient of $\frac{dy}{dx}$ zero, *i.e.*, $P_1 = 0$

i.e., $$\frac{d^2z}{dx^2} + P\frac{dz}{dx} = 0 \quad \text{or} \quad \frac{\frac{d^2z}{dx^2}}{\frac{dz}{dx}} = -P$$

Integrating, $$\log\frac{dz}{dx} = -\int P dx \quad \text{or} \quad \frac{dz}{dx} = e^{-\int P\, dx}$$

Then the equation (2) is reduced to

$$\frac{d^2y}{dz^2} + Q_1 y = R_1$$

which can be solved easily provided Q_1 comes out to be a constant or a constant multiplied by $\frac{1}{z^2}$.

Again, if we choose z such that

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = a^2 \text{ (constant)}$$

i.e., $$a^2\left(\frac{dz}{dx}\right)^2 = Q \quad \text{or} \quad a\frac{dz}{dx} = \sqrt{Q}$$

$$\therefore \quad az = \int \sqrt{Q}\, dx$$

Then equation (2) is reduced to

$$\frac{d^2y}{dz^2} + P_1\frac{dx}{dz} + a^2y = R_1$$

which can be solved easily provided P_1 comes out to be a constant.

Note. It is advised to remember the equation (2) and the values of P_1, Q_1 and R_1.

SOLVED EXAMPLES

Example 51. *Solve :* $\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + \frac{a^2}{x^4}y = 0.$

Sol. Choose z such that

$$\left(\frac{dz}{dx}\right)^2 = Q = \frac{a^2}{x^4}$$

$$\therefore \quad \frac{dz}{dx} = \pm\frac{a}{x^2}, \; z = \pm\frac{a}{x}$$

Changing the independent variable from x to z by the relation $z = \frac{a}{x}$, we get

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1y = R_1$$

where $$P = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{\frac{2a}{x^3} + \frac{2}{x}\left(-\frac{a}{x^2}\right)}{\left(-\frac{a}{x^2}\right)^2} = 0$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} \qquad \text{and} \qquad R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = 0$$

$\therefore$ The transformed equation is $\frac{d^2y}{dz^2} + y = 0$

$$\therefore \qquad y = c_1 \cos z + c_2 \sin z = c_1 \cos \frac{a}{x} + c_2 \sin \frac{a}{x}.$$

Example 52. *Solve :* $x \frac{d^2y}{dx^2} - \frac{dy}{dx} + 4x^3y = x^5$.

Sol. The given equation can be written as

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} + 4x^2y = x^4$$

Choosing z, such that $\left(\frac{dz}{dx}\right)^2 = Q = 4x^2$ or $\frac{dz}{dx} = 2x \quad \therefore \quad z = x^2$

Now changing the independent variable from x to z by the relation $z = x^2$, we get

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1$$

where $$P_1 = \frac{\frac{d^2y}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{2 + \left(-\frac{1}{x}\right)2x}{(2x)^2} = 0$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = 1 \qquad \text{and} \qquad R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{x^4}{(2x)^2} = \frac{x^2}{4} = \frac{1}{4}z$$

The given equation is transformed to

$$\frac{d^2y}{dz^2} + y = \frac{1}{4}z$$

whose $$\text{C.F.} = c_1 \cos z + c_2 \sin z$$

and $$\text{P.I.} = \frac{1}{4} \cdot \frac{1}{D^2+1} z = \frac{1}{4}(1 + D^2)^{-1} z = \tfrac{1}{4}(1 - D^2 + D^4 \,.....) z = \tfrac{1}{4} z$$

$$\therefore \qquad y = c_1 \cos z + c_2 \sin z + \tfrac{1}{4} z$$

or $$y = c_1 \cos x^2 + c_2 \sin x^2 + \tfrac{1}{4} x^2.$$

Example 53. *Solve :* $\frac{d^2y}{dx^2} + \cot x \frac{dy}{dx} + 4y \operatorname{cosec}^2 x = 0$.

Sol. Choosing z, such that

$$\left(\frac{dz}{dx}\right)^2 = 4 \operatorname{cosec}^2 x, \text{ so that}$$

$$\frac{dz}{dx} = 2 \operatorname{cosec} x \text{ or } z = 2 \log \tan \frac{x}{2}.$$

NOTES

Now changing the independent variable from x to z by the relation

$z = 2 \log \tan \frac{x}{2}$, we get

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1$$

where $$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = -\frac{2 \text{ cosec } x \cot x + 2 \cot x \text{ cosec } x}{(2 \text{ cosec } x)^2} = 0$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = 1 \quad \text{and} \quad R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = 0$$

∴ The given equation is transformed to

$$\frac{d^2y}{dz^2} + y = 0$$

∴ $$y = c_1 \cos z + c_2 \sin z \quad \text{or} \quad y = k_1 \cos (z + k_2)$$

or $$y = k_1 \cos \left(2 \log \tan \frac{1}{2} x + k_2\right).$$

Example 54. *Solve:* $(1 + x^2)^2 \frac{d^2y}{dx^2} + 2x(1 + x^2) \frac{dy}{dx} + 4y = 0.$

Sol. The given equation can be written as

$$\frac{d^2y}{dx^2} + \frac{2x}{1+x^2}\frac{dy}{dx} + \frac{4}{(1+x)^2} y = 0$$

Choosing z, such that

$$\left(\frac{dz}{dx}\right)^2 = Q = \frac{4}{1+x^2} \quad \therefore \quad \frac{dz}{dx} = \frac{x}{1+x^2}$$

or $$z = 2 \tan^{-1} x$$

Changing the independent variable from x to z by the relation $z = 2 \tan^{-1} x$, we get

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1$$

where $$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{\frac{-4x}{(1-x^2)^2} + \frac{2x}{1+x^2} \cdot \frac{2}{1+x^2}}{\frac{4}{(1+x^2)^2}} = 0$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = 1 \quad \text{and} \quad R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = 0$$

∴ The transformed equation is $\frac{d^2y}{dx^2} + y = 0$

∴ $$y = c_1 \cos z + c_2 \sin z$$

or $$y = c_1 \cos (2 \tan^{-1} x) + c_2 \sin (2 \tan^{-1} x)$$

$$= c_1 \cos\left(\tan^{-1}\frac{2x}{1-x^2}\right) + c_2 \sin\left(\tan^{-1}\frac{2x}{1-x^2}\right)$$

$$= c_1 \frac{1-x^2}{1+x^2} + c_2 \frac{2x}{1+x^2}$$

or $$y(1 + x^2) = c_1 (1 - x^2) + 2c_2 x.$$

Example 55. *Solve* $x^6 \dfrac{d^2y}{dx^2} + 3x^5 \dfrac{dy}{dx} + a^2y = \dfrac{1}{x^2}$.

Sol. The given equation can be written as

$$\frac{d^2y}{dx^2} + \frac{3}{x}\frac{dy}{dx} + \frac{a^2}{x^6} y = \frac{1}{x^8}$$

Choosing z, such that $$\left(\frac{dz}{dx}\right)^2 = Q = \frac{a^2}{x^6}$$

$\therefore$ $$\frac{dz}{dx} = \frac{a}{x^3} \quad \text{or} \quad z = -\frac{a}{2x^2}.$$

Changing the independent variable from x to z by the relation $z = -\dfrac{a}{2x^2}$, we get

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1$$

where $$P_1 = \frac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2} = 0, \quad Q_1 = \frac{Q}{\left(\dfrac{dz}{dx}\right)^2} = 1$$

and $$R_1 = \frac{R}{\left(\dfrac{dz}{dx}\right)^2} = \frac{1}{a^2x^2} = -\frac{2z}{a^3}$$

$\therefore$ The transformed equation is

$$\frac{d^2y}{dz^2} + y = -\frac{2z}{a^3} \quad \text{whose C.F.} = c_1 \cos z + c_2 \sin z$$

$$= c_1 \cos\left(-\frac{a}{2x^2}\right) + c_2 \sin\left(-\frac{a}{2x^2}\right)$$

$$= c_1 \cos \frac{a}{2x^2} + c_2 \sin \frac{a}{2x^2}$$

and $$\text{P.I.} = \frac{1}{D^2+1}\left(-\frac{2z}{a^3}\right) = -\frac{2}{a^3}(1 + D^2)^{-1} z$$

$$= -\frac{2}{a^3}(1 - D^2 + D^4 \ldots) z = -\frac{2z}{a^3} = \frac{1}{a^2x^2}$$

$\therefore$ $$y = c_1 \cos\frac{a}{2x^2} + c_2 \sin\frac{a}{2x^2} + \frac{1}{a^2x^2}.$$

Example 56. *Solve :* $\cos x \dfrac{d^2y}{dx^2} + \sin x \dfrac{dy}{dx} - 2y \cos^3 x = 2 \cos^5 x.$

Sol. The given equation can be written as

$$\frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} - (2\cos^2 x)\, y = 2\cos^5 x$$

NOTES

Choosing z such that $\left(\frac{dz}{dx}\right)^2 = \cos^2 x$

$$\therefore \quad \frac{dz}{dx} = \cos x \quad \text{or} \quad z = \sin x$$

Changing the independent variable from x to z by the relation $z = \sin x$, we have

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1$$

where $$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{-\sin x + \tan x \cos x}{\cos^2 x} = 0, \qquad Q_1 = \frac{1}{\left(\frac{dz}{dx}\right)^2} = -2$$

and $$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = 2\cos^2 x = 2(1 - z^2)$$

$\therefore$ The transformed equation is

$$\frac{d^2y}{dz^2} - 2y = 2(1 - z^2)$$

whose $$\text{C.F.} = c_1 e^{\sqrt{2}z} + c_2 e^{-\sqrt{2}z}$$

and $$\text{P.I.} = \frac{1}{D^2 - 2} \cdot 2(1 - z^2) = -\left(1 - \frac{D^2}{2}\right)^{-1} (1 - z^2)$$

$$= -\left(1 + \frac{D^2}{2} + \frac{D^4}{4} \ldots\ldots\right)(1 - z^2)$$

$$= -(1 - z^2) + \tfrac{1}{2}(+2) = z^2$$

$$\therefore \quad y = c_1 e^{\sqrt{2}z} + c_2 e^{-\sqrt{2}z} + z^2$$

Required solution is $$= c_1 e^{\sqrt{2}\sin x} + c_2 e^{-\sqrt{2}\sin x} + \sin^2 x.$$

Example 57. *Solve:* $\frac{d^2y}{dx^2} + \left(1 - \frac{1}{x}\right)\frac{dy}{dx} + 4x^2 e^{-2x} y = 4(x^2 + x^3)\, e^{-3x}.$

Sol. Choosing z, such that

$$\left(\frac{dz}{dx}\right)^2 = 4x^2 e^{-2x} \qquad \therefore \qquad \frac{dz}{dx} = 2xe^{-x} \quad \text{or} \quad z = -2(x + 1)\, e^{-x}.$$

Changing the independent variable from x to z by the relation,
$x = -2(x + 1)\, e^{-x}$, we have

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dx} + Q_1 y = R_1$$

where $$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{2(1 - x)\, e^{-x} + \left(1 - \frac{1}{x}\right) 2xe^{-x}}{4x^2 e^{-2x}}$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = 1 \qquad \text{and} \qquad R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = (1 + x)\, e^{-x} = -\tfrac{1}{2} z$$

∴ Transformed equation is

$$\frac{d^2y}{dz^2} + y = -\frac{1}{2}z$$

whose C.F. $= c_1 \cos z + c_2 \sin z$

and P.I. $= \dfrac{1}{D^2+1}\left(-\dfrac{1}{2}z\right) = -\dfrac{1}{2}(1 + D^2)^{-1}z$

$$= -\tfrac{1}{2}(1 - D^2 + D^4 - .....)\, z = -\tfrac{1}{2}z$$

∴ $y = c_1 \cos z + c_2 \sin z - \frac{1}{2}z$

$$y = c_1 \cos \{2(x+1)e^{-x}\} - c_2 \sin \{2(x+1)\, e^{-x}\} + (x+1)\, e^{-x}.$$

Example 58. *Solve :* $x\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 4x^3y = 8x^3 \sin x^2.$

Sol. The given equation can be written as

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} - 4x^2y = 8x^2 \sin x^2$$

Choosing z, such that

$$\left(\frac{dz}{dx}\right)^2 = 4x^2 \quad \text{or} \quad \frac{dz}{dx} = 2x \quad \therefore \quad z = x^2$$

Changing the independent variable from x to z by the relation $z = x^2$, we get

$$\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + Q_1y = R_1$$

where $P_1 = \dfrac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2} = 0,\ Q_1 = \dfrac{Q}{\left(\dfrac{dz}{dx}\right)^2} = -1$

and $R_1 = \dfrac{R}{(dz/dx)^2} = 2 \sin x^2 = 2 \sin z$

∴ The transformed equation is

$$d^2y/dz^2 - y = 2 \sin z$$

whose C.F. $= c_1e^z + c_2e^{-z}$

and P.I. $= \dfrac{1}{D^2 - 1}(2 \sin z) = \dfrac{1}{-1^2 - 1} \sin z = -\sin z$

∴ $y = c_1e^z + c_2e^{-z} - \sin z$

∴ The solution of the given equation is

$$y = c_1e^{x^2} + c_2e^{-x^2} - \sin x^2.$$

EXERCISE G

Solve the following differential equations:

1. $x\dfrac{d^2y}{dx^2} + (4x^2 - 1)\dfrac{dy}{dx} + 4x^3y = 2x^3$

2. $x^4\dfrac{d^2y}{dx^2} + 2x^3\dfrac{dy}{dx} + n^2y = 0$

NOTES

3. $\frac{d^2y}{dx^2} - \cot x \frac{dy}{dx} - y \sin^2 x = 0$

4. $\frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} + y \cos^2 x = 0$

5. $(1 + x)^2 \frac{d^2y}{dx^2} + (1 + x)\frac{dy}{dx} + y = 4 \cos [\log(1 + x)]$

6. $\frac{d^2y}{dx^2} + (3 \sin x - \cot x)\frac{dy}{dx} + 2y \sin^2 x = e^{-\cos x} \sin^2 x$

7. $\frac{d^2y}{dx^2} - \cot x \frac{dy}{dx} - y \sin^2 x = \cos x - \cos^3 x$

8. $\frac{d^2y}{dx^2} + (\tan x - 3 \cos x)\frac{dy}{dx} + 2y \cos^2 x = \cos^4 x$

9. $y'' - (1 + 4e^x)y' + 3e^{2x}y = e^{2(x + e^x)}$

10. $y'' - (8e^{2x} + 2)y' + 4e^{4x}y = e^{6x}$

Answers

1. $y = e^{-x^2}(c_1x^2 + c) + \frac{1}{2}$

2. $y = c \cos\left(\frac{n + \alpha x}{x}\right)$

3. $y = c_1e^{-\cos x} + c_2e^{\cos x}$

4. $y = c_1 \sin (\sin x + c_2)$

5. $y = c_1 \cos \log(1 + x) + c_2 \sin \log(1 + x) + 2 \log(1 + x) \sin \log(1 + x)$

METHOD OF VARIATION OF PARAMETERS

Here we shall explain the method of finding the complete primitive of a linear equation whose complimentary function is known.

Let $y = A\phi(x) + B\psi(x)$ be the complimentary function of the linear equation of second order

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad ...(1)$$

where A and B are constants and $\phi(x)$ and $\psi(x)$ are functions of x

Since, $y = A\phi(x) + B\psi(x)$

Satisfies the equation $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$

$$\therefore \quad [A\phi''(x) + B\psi''(x)] + P[A\phi'(x) + B\psi'(x)] + Q[A\phi(x) + B\psi(x)] = 0$$

or
$$A[\phi''(x) + P\phi'(x) + Q\phi(x)] + B[\psi''(x) + P\psi'(x) + Q\psi(x)] = 0$$

$$\therefore \quad \phi''(x) + P\phi'(x) + Q\phi(x) = 0 \quad ...(2)$$

and
$$\psi''(x) + P\psi'(x) + Q\psi(x) = 0 \quad ...(3)$$

Now let us assume that

$$y = A\phi(x) + B\psi(x) \quad ...(4)$$

is the complete primitive of (1) where A and B are not constants but functions of x to be so chosen that (1) will be satisfied.

$$\therefore \quad \frac{dy}{dx} = A\phi'(x) + B\psi'(x) + \frac{dA}{dx}\phi(x) + \frac{dB}{dx}\psi(x)$$

Let A and B satsify the equation,

$$\phi(x)\frac{dA}{dx} + \psi(x)\frac{dB}{dx} = 0 \quad ...(5)$$

$\therefore \qquad \dfrac{dy}{dx} = A\phi'(x) + B\psi'(x)$

and $\qquad \dfrac{d^2y}{dx^2} = A\phi''(x) + B\psi''(x) + \dfrac{dA}{dx}\phi'(x) + \dfrac{dB}{dx}\psi'(x).$

Substituting in (1), we have

$$\left[A\phi''(x) + B\psi''(x) + \frac{dA}{dx}\phi'(x) + \frac{dB}{dx}\psi'(x)\right] + P\,[A\phi'(x) + B\psi'(x)] + Q[A\phi(x) + B\psi(x)] = R$$

or $\qquad A[\phi''(x) + P\phi'(x) + Q\phi(x)] + B[\psi''(x) + P\psi'(x) + Q\psi(x)] + \phi'(x)\dfrac{dA}{dx} + \psi'(x)\dfrac{dB}{dx} = R$

Since the co-efficient of A and B are zero [by (2) and (3),] we have

$$\phi'(x)\frac{dA}{dx} + \psi'(x)\frac{dB}{dx} = R \qquad ...(6)$$

Solving (5) and (6) for $\dfrac{dA}{dx}$ and $\dfrac{dB}{dx}$, we get

$$\frac{dA}{dx} = \frac{-R\,\psi(x)}{\phi(x)\,\psi'(x) - \phi'(x)\,\psi(x)} = \frac{-R\,\psi(x)}{W}$$

and $$\frac{dB}{dx} = \frac{R\,\phi(x)}{\phi(x)\,\psi'(x) - \phi'(x)\,\psi(x)} = \frac{R\,\phi(x)}{W}$$

where $W = \begin{vmatrix} \phi(x) & \psi(x) \\ \phi'(x) & \psi'(x) \end{vmatrix}$ is called the Wronskian of $\phi(x)$ and $\psi(x)$.

Integrating (7), $\qquad A = -\int \dfrac{R\,\psi(x)}{W}dx + c_1, \; B = \int \dfrac{R\,\phi(x)}{W}dx + c_2$

Substituting these values of A and B in (4), we get the complete solution of (1).

Note 1. As the solution is obtained by varying the arbitrary constants of the complementary function, the method is known as variation of parameters.

2. Method of variation of parameters is to be used if instructed to do so.

SOLVED EXAMPLES

Example 59. *Apply the method of variation of parameters to solve:*

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2e^x.$$

Sol. The given equation in standard form is

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = e^x \qquad ...(1)$$

Here $\quad P = \dfrac{1}{x}, \quad Q = -\dfrac{1}{x^2}, \quad R = e^x$

Now to find the C.F. of (1) *i.e.*, the solution of the equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = 0$$

NOTES

or $$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0 \quad ...(2)$$

which is a homogeneous equation, put $x = e^z$ so that $z = \log x$.

Let $D \equiv \frac{d}{dz}$, then equation (2) becomes

$$[D(D-1)+D-1]y=0 \quad \text{or} \quad (D^2-1)y=0$$

Its A.E. is $m^2 - 1 = 0$ so that $m = \pm 1$

∴ Solution of (2) is $y = c_1e^z + c_2e^{-z} = c_1x + c_2x^{-1}$

⇒ Parts of C.F. of (1) are $\phi(x) = x$ and $\psi(x) = \frac{1}{x}$

Wronskian of $\phi(x)$ and $\psi(x)$ is

$$W=\begin{vmatrix}\phi(x) & \psi(x)\\ \phi'(x) & \psi'(x)\end{vmatrix}=\begin{vmatrix}x & \frac{1}{x}\\ 1 & -\frac{1}{x^2}\end{vmatrix}=-\frac{1}{x}-\frac{1}{x}=-\frac{2}{x}$$

Let $y = A\phi(x) + B\psi(x) = Ax + \frac{B}{x}$ be the complete solution of (1) where A and B are functions of x determined as follows:

$$A=-\int\frac{R\,\psi(x)}{W}dx+c_1=-\int\frac{\frac{1}{x}\cdot e^x}{-\frac{2}{x}}dx+c_1=\frac{1}{2}\int e^x\,dx+c_1=\frac{1}{2}e^x+c_1$$

and $$B=\int\frac{R\,\phi(x)}{W}dx+c_2=\int\frac{e^x\cdot x}{-\frac{2}{x}}dx+c_2$$

$$=-\frac{1}{2}\int x^2e^xdx+c_2=-\frac{1}{2}\left(x^2-2x+2\right)e^x+c_2=-\frac{1}{2}x^2e^x+(x-1)\,e^x+c_2$$

Hence, the complete solution of (1) is

$$y=A\phi(x)+B\psi(x)$$

$$=\left(\frac{1}{2}e^x+c_1\right)x+\left[-\frac{1}{2}x^2e^x+(x-1)e^x+c_2\right]\cdot\frac{1}{x}$$

$$=c_1x+\frac{c_2}{x}+\frac{1}{2}xe^x-\frac{1}{2}xe^x+\left(\frac{x-1}{x}\right)e^x$$

or $$y=c_1x+\frac{c_2}{x}+\left(1-\frac{1}{x}\right)e^x$$

where c_1 and c_2 are arbitrary constants of integration.

Example 60. *Using method of variation of parameters, solve:*

$$x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-12y=x^3\log x.$$

Sol. The given equation in standard form is

$$\frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}-\frac{12}{x^2}y=x\log x \quad ...(1)$$

Here $P = \frac{2}{x}$, $Q = -\frac{12}{x^2}$, $R = x \log x$

Now to find the C.F. of (1) *i.e.*, the solution of the equation

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} - \frac{12}{x^2}y = 0$$

or $$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 12y = 0 \qquad \ldots(2)$$

which is a homogeneous equation, put $x = e^z$ so that $z = \log x$.

Let $D \equiv \frac{d}{dz}$, the equation (2) becomes

$$[D(D-1) + 2D - 12]y = 0 \quad \text{or} \quad (D^2 + D - 12)y = 0$$

Its A.E. is $m^2 + m - 12 = 0$ or $(m + 4)(m - 3) = 0$

$\Rightarrow$ $m = 3, -4$

$\therefore$ Solution of (2) is $y = c_1e^{3z} + c_2e^{-4z} = c_1x^3 + c_2x^{-4}$

$\Rightarrow$ Parts of C.F. of (1) are $\phi(x) = x^3$ and $\psi(x) = \frac{1}{x^4}$

Wronskian of $\phi(x)$ and $\psi(x)$ is

$$W = \begin{vmatrix} \phi(x) & \psi(x) \\ \phi'(x) & \psi'(x) \end{vmatrix} = \begin{vmatrix} x^3 & \frac{1}{x^4} \\ 3x^2 & -\frac{4}{x^5} \end{vmatrix} = -\frac{4}{x^2} - \frac{3}{x^2} = -\frac{7}{x^2}.$$

Let $y = A\phi(x) + B\psi(x) = Ax^3 + \frac{B}{x^4}$ be the complete solution of (1) where A and B are functions of x determined as follows:

$$A = -\int \frac{R\,\psi(x)}{W}dx + c_1 = -\int \frac{x \log x \,.\, \frac{1}{x^4}}{-\frac{7}{x^2}}dx + c_1$$

$$= \frac{1}{7}\int (\log x)\,.\,\frac{1}{x}\,dx + c_1 = \frac{1}{7}\,.\,\frac{(\log x)^2}{2} + c_1 = \frac{1}{14}(\log x)^2 + c_1$$

and $$B = \int \frac{R\,\phi(x)}{W}dx + c_2 = \int \frac{x \log x\,.\,x^3}{-\frac{7}{x^2}}dx + c_2$$

$$= -\frac{1}{7}\int (\log x)\,.\,x^6 dx + c_2 = -\frac{1}{7}\left[(\log x)\,.\,\frac{x^7}{7} - \int \frac{1}{x}\,.\,\frac{x^7}{7}dx\right] + c_2$$

$$= -\frac{x^7}{49}\log x + \frac{1}{49}\,.\,\frac{x^7}{7} + c_2 = \frac{x^7}{49}\left(-\log x + \frac{1}{7}\right) + c_2$$

$$y = A\,\phi(x) + B\,\psi(x)$$

$$= \left[\frac{1}{14}(\log x)^2 + c_1\right]x^3 + \left[\frac{x^7}{49}\left(-\log x + \frac{1}{7}\right) + c_2\right]\cdot\frac{1}{x^4}$$

$$= c_1x^3 + \frac{c_2}{x^4} + \frac{x^3}{14}(\log x)^2 + \frac{x^3}{49}\left(-\log x + \frac{1}{7}\right).$$

Example 61. *Apply the method of variation of parameters to solve:*

$$x^2 \frac{d^2y}{dx^2} - 2x(1+x)\frac{dy}{dx} + 2(1+x)y = x^3.$$

Sol. The given equation in standard form is

$$\frac{d^2y}{dx^2} - \frac{2(1+x)}{x}\frac{dy}{dx} + \frac{2(1+x)}{x^2}y = x \qquad \ldots(1)$$

Here, $\quad P = -\frac{2(1+x)}{x}, \quad Q = \frac{2(1+x)}{x^2}, R = x$

Since $\quad P + Qx = -\frac{2(1+x)}{x} + \frac{2(1+x)}{x} = 0$

$\therefore \quad y = x$ is a part of C.F.

Now to find the C.F. of (1), *i.e.*, the solution of the equation

$$\frac{d^2y}{dx^2} - \frac{2(1+x)}{x}\frac{dy}{dx} + \frac{2(1+x)}{x^2}y = 0 \qquad \ldots(2)$$

Put $\quad y = vx$ so that $\frac{dy}{dx} = \frac{dv}{dx}x + v$ and $\frac{d^2y}{dx^2} = \frac{d^2v}{dx^2}x + 2\frac{dv}{dx}$

Substituting in (2), we have

$$x\frac{d^2v}{dx^2} + 2\frac{dv}{dx} - \frac{2(1+x)}{x}\left(x\frac{dv}{dx} + v\right) + \frac{2(1+x)}{x^2} \cdot vx = 0$$

or $$x\frac{d^2v}{dx^2} + 2\frac{dv}{dx} - 2(1+x)\frac{dv}{dx} - \frac{2(1+x)}{x}v + \frac{2(1+x)}{x}v = 0$$

or $$x\frac{d^2v}{dx^2} - 2x\frac{dv}{dx} = 0 \quad \text{or} \quad \frac{d^2v}{dx^2} - 2\frac{dv}{dx} = 0$$

Its A.E. is $\quad m^2 - 2m = 0 \quad$ so that $m = 0, 2$

$\therefore \quad v = c_1 e^{0x} + c_2 e^{2x} = c_1 + c_2 e^{2x}$

$\Rightarrow$ Solution of (2) is $y = vx = c_1 x + c_2 x\, e^{2x}$

$\Rightarrow$ Parts of C.F. of (1) are $\phi(x) = x$ and $\psi(x) = xe^{2x}$

Wronskian of $\phi(x)$ and $\psi(x)$ is

$$W = \begin{vmatrix} \phi(x) & \psi(x) \\ \phi'(x) & \psi'(x) \end{vmatrix} = \begin{vmatrix} x & xe^{2x} \\ 1 & (1+2x)e^{2x} \end{vmatrix} = 2x^2 e^{2x}$$

Let $\quad y = A\,\phi(x) + B\psi(x) = Ax + Bxe^{2x}$ be the complete solution of (1) where A and B are functions of x determined as follows:

$$A = -\int \frac{R\,\psi(x)}{W}dx + c_1 = -\int \frac{x\,.\,xe^{2x}}{2x^2\,e^{2x}}dx + c_1$$

$$= -\frac{1}{2}\int dx + c_1 = -\frac{x}{2} + c_1$$

and $$B = \int \frac{R\,\phi(x)}{W}dx + c_2 = \int \frac{x\,.\,x}{2x^2 e^{2x}}dx + c_2 = \frac{1}{2}\int e^{-2x}\,dx + c_2 = -\frac{1}{4}e^{-2x} + c_2$$

Hence, the complete solution of (1) is

$$y = A\,\phi(x) + B\psi(x) = \left(-\frac{x}{2} + c_1\right)x + \left(-\frac{1}{4}e^{-2x} + c_2\right)xe^{2x}$$

or $$y = c_1 x + c_2 xe^{2x} - \frac{x^2}{2} - \frac{x}{4}$$

where c_1 and c_2 are arbitrary constants of integration.

Example 62. *Using method of variation of parameters, solve*

$$(1-x)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = (1-x)^2.$$

Sol. The given equation in standard form is

$$\frac{d^2y}{dx^2} + \frac{x}{1-x}\cdot\frac{dy}{dx} - \frac{1}{1-x}y = 1-x \qquad ...(1)$$

Here, $\quad P = \frac{x}{1-x}, \quad Q = -\frac{1}{1-x}, \quad R = 1-x$

Since, $\quad P + Qx = \frac{x}{1-x} - \frac{x}{1-x} = 0$

$\therefore \quad y = x$ is a part of C.F.

Now to find the C.F. of (1), *i.e.*, the solution of the equation

$$\frac{d^2y}{dx^2} + \frac{x}{1-x}\frac{dy}{dx} - \frac{1}{1-x}y = 0 \qquad ...(2)$$

Put $y = vx$ so that $\quad \frac{dy}{dx} = \frac{dv}{dx}x + v \quad$ and $\quad \frac{d^2y}{dx^2} = \frac{d^2v}{dx^2}x + 2\frac{dv}{dx}$

Substituting in (2), we have

$$x\frac{d^2v}{dx^2} + 2\frac{dv}{dx} + \frac{x}{1-x}\left(x\frac{dv}{dx} + v\right) - \frac{1}{1-x}.vx = 0$$

or $$x\frac{d^2v}{dx^2} + 2\frac{dv}{dx} + \frac{x^2}{1-x}\frac{dv}{dx} = 0$$

or $$\frac{d^2v}{dx^2} + \left(\frac{2}{x} + \frac{x}{1-x}\right)\frac{dv}{dx} = 0$$

or $$\frac{dp}{dx} + \left(\frac{2}{x} + \frac{1}{1-x} - 1\right)p = 0 \quad \text{where } p = \frac{dv}{dx}$$

or $$\frac{dp}{p} = \left(1 - \frac{1}{1-x} - \frac{2}{x}\right)dx$$

Integrating, $\quad \log p = x + \log(1-x) - 2\log x + \log c_1$

$$= \log\frac{c_1(1-x)e^x}{x^2}$$

or $$p = \frac{c_1(1-x)e^x}{x^2} \quad \text{or} \quad \frac{dv}{dx} = \frac{c_1(1-x)e^x}{x^2}$$

or $$dv = c_1\left(\frac{1}{x^2} - \frac{1}{x}\right)e^x\,dx \qquad |\ \text{Form } [f'(x) + f(x)]e^x$$

Integrating, $$v = c_1\left(-\frac{1}{x}\right)e^x + c_2 \qquad |\ f(x)e^x$$

$\therefore \quad$ Solution of (2) is $\quad y = vx = -c_1e^x + c_2x$

$\Rightarrow \quad$ Parts of C.F. of (1) are $\phi(x) = -e^x$ and $\psi(x) = x$

Wronskian of $\phi(x)$ and $\psi(x)$ is $\quad W = \begin{vmatrix} \phi(x) & \psi(x) \\ \phi'(x) & \psi'(x) \end{vmatrix} = \begin{vmatrix} -e^x & x \\ -e^x & 1 \end{vmatrix} = (x-1)e^x$

NOTES

Let $y = A\phi(x) + B\psi(x) = A(-e^x) + Bx$ be the complete solution of (1) where A and B are functions of x determined as follows:

$$A = -\int \frac{R\,\psi(x)}{W}\,dx + c_1 = -\int \frac{(1-x)x}{(x-1)e^x}\,dx + c_1 = \int xe^{-x}\,dx + c_1$$

$$= x(-e^{-x}) - \int 1.(-e^{-x})\,dx + c_1 = -xe^{-x} - e^{-x} + c_1$$

$$= -(x+1)e^{-x} + c_1$$

and

$$B = \int \frac{R\,\phi(x)}{W}\,dx + c_2 = \int \frac{(1-x)(-e^x)}{(x-1)e^x}\,dx + c_2 = \int dx + c_2$$

$$= x + c_2$$

Hence, the complete solution of (1) is

$$y = A\phi(x) + B\psi(x)$$

$$= [-(x+1)e^{-x} + c_1](-e^x) + (x + c_2)x$$

or

$$y = -c_1e^x + c_2x + x + 1 + x^2$$

where c_1 and c_2 are arbitrary constants of integration.

EXERCISE H

Using method of variation of parameters, solve the following differential equations:

1. $x^2\dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = e^x$

2. $x^2\dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = x^2 + \dfrac{1}{x^2}$

3. $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - 9y = 48\,x^5$

4. $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = x\log x$

5. $x^2\dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + 6y = \sin(\log x)$

6. $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = x^3e^x$

7. $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = x^2\log x$

8. $(1-x)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = 2(1-x)^2\,e^{-x}$

Answers

1. $y = \dfrac{c_1}{x} + \dfrac{c_2}{x^2} + \dfrac{1}{x^2}e^x$

2. $y = \dfrac{c_1}{x} + \dfrac{c_2}{x^2} + \dfrac{1}{12}x^2 - \dfrac{1}{x^2}\log x$

3. $y = c_1\,x^3 + c_2x^{-3} + 3x^5$

4. $y = c_1\,x\log x + c_2x + \dfrac{x}{6}(\log x)^3$

5. $y = c_1\,x^2 + c_2x^3 + \dfrac{1}{10}\,[(\sin(\log x) + \cos(\log x)]$

6. $y = c_1 + c_2\,x^2 + (x-1)e^x$

7. $y = c_1x + \dfrac{c_2}{x} + \dfrac{1}{3}x^3\log x - \dfrac{4}{9}x^2$

8. $y = c_1x + c_2e^x + \left(\dfrac{1}{2} - x\right)e^{-x}$

UNIT IV

5. POWER SERIES SOLUTIONS

NOTES

STRUCTURE

Introduction
Definitions
Power series solution, when $x = 0$ is an ordinary point of the equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

Frobenius Method: Series solution when x = 0 is a regular singular point of the differential equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

INTRODUCTION

The solution of ordinary linear differential equations of second order with variable coefficients in the form of an infinite convergent series is called *solution in series* or *integration in series*.

The series solution of certain differential equations give rise to *special functions* such as Bessel's function, Legendre's polynomials, Laguerre's polynomial, Hermite's polynomial, Chebyshev polynomials. These special functions have wide applications in engineering.

In this unit, we will discuss methods of solution of second order linear differential equations with variable coefficients in series along with Bessel's function, Legendre's polynomial and their properties.

DEFINITIONS

Power Series

An infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \cdots$$

is called a power series in ascending powers of $x - x_0$.

In particular, a power series in ascending powers of x is an infinite series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

e.g.,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Analytic Function

A function $f(x)$ defined on an interval containing the point $x = x_0$ is called analytic at x_0 if its Taylor series $\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$ exists and converges to $f(x)$ for all x in the interval of convergence of Taylor's series.

Note 1. A rational function is analytic except at those values of x at which its denominator is zero. *e.g.,* Rational function $\frac{x}{x^2 - 5x + 6}$ is analytic everywhere except at $x = 2$ and $x = 3$.

Note 2. All polynomial functions e^x, $\sin x$, $\cos x$, $\sinh x$ and $\cosh x$ are analytic everywhere.

Ordinary Point

A point $x = x_0$ is called an ordinary point of the equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\, y = 0 \qquad \ldots(1)$$

if both the functions $P(x)$ and $Q(x)$ are analytic at $x = x_0$.

Regular and Irregular Singular Points

If the point $x = x_0$ is not an ordinary point of the differential equation (1), then it is called a singular point of equation (1). There are two types of singular points:

(*i*) Regular singular point.

(*ii*) Irregular singular point.

A singular point $x = x_0$ of the differential equation (1) is called a regular singular point of (1) if both $(x - x_0)\, P(x)$ and $(x - x_0)^2\, Q(x)$ are analytic at $x = x_0$.

A singular point which is not regular is called an irregular singular point.

Remark 1. When $x = 0$ is an ordinary point of equation (1), its every solution can be expressed as a series of the form

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{n=0}^{\infty} a_n x^n .$$

Remark 2. When $x = 0$ is a regular singular point of equation (1), at least one of its solution can be expressed as

$$y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \cdots = x^m (a_0 + a_1 x + a_2 x^2 + \cdots) = \sum_{n=0}^{\infty} a_n x^{m+n}$$

where m may be a positive or negative integer or a fraction.

Remark 3. If $x = 0$ is an irregular singular point of equation (1), then discussion of solution of the equation is beyond the scope of this book.

NOTES

POWER SERIES SOLUTION, WHEN $x = 0$ IS AN ORDINARY POINT OF THE EQUATION

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

Steps for solution:

1. Assume its solution to be of the form $y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$ ...(1)
2. Find $\frac{dy}{dx}$ (or y') and $\frac{d^2y}{dx^2}$ (or y'') from y.
3. Substitute the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given differential equation.
4. Equate to zero the coefficients of various powers of x and find $a_2, a_3, a_4, a_5, \cdots$ in terms of a_0 and a_1.
5. Equate to zero, the coefficient of x^n. The relation so obtained is called the *recurrence relation*. It helps us in finding the values of other constants easily.
6. Give different values to n in the recurrence relation to determine various a_i's in terms of a_0 and a_1.
7. Substitute the values of $a_2, a_3, a_4, \ldots$ in assumed solution (1) above to get the series solution of the given equation having a_0 and a_1 as arbitrary constants.

SOLVED EXAMPLES

Example 1. *Solve in series the differential equation*

$$\frac{d^2y}{dx^2} + xy = 0.$$

Sol. Comparing the given equation with the form $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0$,

we get $\qquad P(x) = 0,\ Q(x) = x$

At $x = 0$, both P(x) and Q(x) are analytic, hence $x = 0$ is an *ordinary point*.

Assume its solution to be

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots \qquad ...(1)$$

Then, $$\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots + n\,a_nx^{n-1} + \cdots$$

and $$\frac{d^2y}{dx^2} = 2\cdot1\,a_2 + 3\cdot2\,a_3x + 4\cdot3\,a_4x^2 + 5\cdot4\,a_5x^3 + \cdots + n(n-1)\,a_nx^{n-2} + \cdots$$

Substituting these values in the given differential equation, we get

$$[2\cdot1\cdot a_2 + 3\cdot2\cdot a_3x + 4\cdot3\cdot a_4x^2 + 5\cdot4\cdot a_5x^3 + \cdots + n(n-1)\,a_nx^{n-2} + \cdots] + x\,[a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots] = 0$$

$$2\cdot1\cdot a_2 + (3\cdot2\,a_3 + a_0)\,x + (4\cdot3\cdot a_4 + a_1)x^2 + (5\cdot4\cdot a_5 + a_2)x^3 + \cdots + \{(n+2)(n+1)\,a_{n+2} + a_{n-1}\}x^n + \cdots = 0$$

NOTES

Equating to zero, the various powers of x as,

Coefficient of $x^0 = 0$

$\Rightarrow \quad 2\cdot1 \cdot a_2 = 0 \quad \Rightarrow \quad \boxed{a_2 = 0}$

Coefficient of $x = 0$

$\Rightarrow \quad 3\cdot2 \cdot a_3 + a_0 = 0$

$\Rightarrow \quad a_3 = -\dfrac{a_0}{3\cdot2} \quad \Rightarrow \quad \boxed{a_3 = -\dfrac{a_0}{3!}}$

Coefficient of $x^2 = 0$

$\Rightarrow \quad 4\cdot3 \cdot a_4 + a_1 = 0$

$\Rightarrow \quad a_4 = -\dfrac{a_1}{4\cdot3} \quad$ or $\quad \boxed{a_4 = -\dfrac{2a_1}{4!}}$

Coefficient of $x^3 = 0$

$\Rightarrow \quad 5\cdot4 \cdot a_5 + a_2 = 0$

$\Rightarrow \quad a_5 = -\dfrac{a_2}{5\cdot4} \quad$ or $\quad \boxed{a_5 = 0}$

Coefficient of $x^4 = 0$

$\Rightarrow \quad 6\cdot5 \cdot a_6 + a_3 = 0$

$\Rightarrow \quad a_6 = -\dfrac{a_3}{6\cdot5} = \dfrac{a_0}{6\cdot5\cdot3!} \quad$ or $\quad \boxed{a_6 = \dfrac{4a_0}{6!}}$

and so on.

Coefficient of $x^n = 0$

$\Rightarrow \quad (n + 2)(n + 1)\, a_{n+2} + a_{n-1} = 0$

$$\Rightarrow \quad \boxed{a_{n+2} = -\frac{a_{n-1}}{(n+2)(n+1)}}$$

which is the recurrence relation.

Putting $n = 5, 6, 7, \ldots$, successively in recurrence relation, we obtain

$$a_7 = \frac{5\cdot2a_1}{7!},\ a_8 = 0,\ a_9 = \frac{-7\cdot4}{9!}\ a_0 \text{ and so on.}$$

Substituting these values in (1), we get

$$y = a_0 + a_1x - \frac{a_0}{3!}x^3 - \frac{2a_1}{4!}x^4 + \frac{4a_0}{6!}x^6 + \frac{5.2\,a_1}{7!}x^7 - \frac{7.4}{9!}a_0x^9 + \cdots$$

$$\Rightarrow \quad y = a_0\left[1 - \frac{x^3}{3!} + \frac{1.4}{6!}x^6 - \frac{1.4.7}{9!}x^9 + \ldots\right] + a_1\left[x - \frac{2}{4!}x^4 + \frac{2.5}{7!}x^7 - \ldots\right]$$

where a_0 and a_1 are constants.

Example 2. *Solve in series the differential equation*

$$(1 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0 \text{ about the point } x = 0.$$

Sol. Comparing the given differential equation with the form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0, \text{ we get}$$

$$P(x) = \frac{x}{1+x^2} \quad \text{and} \quad Q(x) = \frac{-1}{1+x^2}.$$

Both P(x) and Q(x) are analytic at $x = 0$

$\therefore$ $x = 0$ is an *ordinary point* of the given differential equation.

Assume the solution to be

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots \quad ...(1)$$

Then, $$\frac{dy}{dx} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + na_n x^{n-1} + \cdots$$

and $$\frac{d^2y}{dx^2} = 2{\cdot}1 \cdot a_2 + 3{\cdot}2 \cdot a_3 x + \cdots + n(n-1)a_n\, x^{n-2} + \cdots$$

Substituting these values in given equation, we get

$$(1 + x^2)\,[2{\cdot}1 \cdot a_2 + 3{\cdot}2 \cdot a_3 x + 4{\cdot}3 \cdot a_4 x^2 + \cdots + n(n-1)\, a_n x^{n-2} + \cdots]$$
$$+ x\,[a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots + n\, a_n x^{n-1} + \cdots]$$
$$- [a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots] = 0$$

0

Coefficient of $x^0 = 0$

$\Rightarrow$ $2.1.\ a_2 - a_0 = 0$ $\Rightarrow$ $\boxed{a_2 = \dfrac{a_0}{2}}$

Coefficient of $x = 0$

$\Rightarrow$ $3.2\ a_3 + a_1 - a_1 = 0$ $\Rightarrow$ $\boxed{a_3 = 0}$

Coefficient of $x^2 = 0$

$\Rightarrow$ $2.1\ .\ a_2 + 4.3.a_4 + 2a_2 - a_2 = 0$

$\Rightarrow$ $4.3\ a_4 + 3a_2 = 0$

$\Rightarrow$ $a_4 = -\dfrac{a_2}{4} = -\dfrac{a_0}{8}$ or $\boxed{a_4 = -\dfrac{a_0}{8}}$

Coefficient of $x^3 = 0$

$\Rightarrow$ $5.4.\ a_5 + 3.2.\ a_3 + 3a_3 - a_3 = 0$

$\Rightarrow$ $20a_5 + 8a_3 = 0$ $\Rightarrow$ $\boxed{a_5 = 0}$

Coefficient of $x^4 = 0$

$\Rightarrow$ $6.5\ .\ a_6 + 4.3\ .\ a_4 + 4a_4 - a_4 = 0$

$\Rightarrow$ $30a_6 + 15a_4 = 0$

$\Rightarrow$ $a_6 = -\dfrac{a_4}{2} = \dfrac{a_0}{16}$ or $\boxed{a_6 = \dfrac{a_0}{16}}$

Similarly, $a_7 = 0$, $a_9 = 0$, $a_{11} = 0$ and so on.

Also, Coefficient of $x^n = 0$

$$(n + 2)\,(n + 1)\, a_{n+2} + n(n-1)a_n + na_n - a_n = 0$$

$\Rightarrow$ $a_{n+2} = -\left(\dfrac{n-1}{n+2}\right) a_n$ $\quad | \because\ n + 1 \neq 0$

Putting $n = 6, 8, 10, \ldots$, we get

$$a_8 = -\frac{5}{8}\, a_6 = -\frac{5a_0}{128}$$

$$a_{10} = -\frac{7}{10}\, a_8 = \frac{7a_0}{256} \text{ and so on.}$$

NOTES

Substituting these values in (1), we get

$$y = a_0 + a_1x + \frac{a_0}{2}x^2 - \frac{a_0}{8}x^4 + \frac{a_0}{16}x^6 - \frac{5a_0}{128}x^8 + \frac{7a_0}{256}x^{10} - \cdots$$

$$\Rightarrow \qquad y = a_0\left(1 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16} - \frac{5x^8}{128} + \frac{7x^{10}}{256} - \cdots\right) + a_1x$$

where a_0 and a_1 are constants.

Example 3. *Solve: $(1 - x^2)y'' - xy' + 4y = 0$ in series.*

Sol. Comparing the given differential equation with the form

$y'' + P(x)\,y' + Q(x)\,y = 0$, we get

$$P(x) = \frac{-x}{1-x^2},\ Q(x) = \frac{4}{1-x^2}$$

Since both P(x) and Q(x) are analytic at $x = 0$, hence $x = 0$ is an *ordinary point* of the given equation.

Assume the solution to be

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots \qquad \ldots(1)$$

Then, $\qquad y' = a_1 + 2.a_2x + 3.a_3x^2 + \cdots + na_nx^{n-1} + \cdots$

and $\qquad y'' = 2.1.\ a_2 + 3.2.a_3x + \cdots + n(n-1)a_n\,x^{n-2} + \cdots$

Substituting these values in given equation, we get

$$(1 - x^2)\ [2.1.\ a_2 + 3.2.\ a_3x + 4.3.\ a_4x^2 + \cdots + n(n-1)\ a_n\,x^{n-2} + \cdots]$$
$$- x\,[a_1 + 2a_2x + 3a_3x^2 + \cdots + na_n\,x^{n-1} + \cdots] + 4\,[a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots]$$
$$= 0$$

Coefficient of $x^0 = 0$

$\Rightarrow \qquad 2.1.\ a_2 + 4a_0 = 0 \qquad \Rightarrow \qquad a_2 = -\,2a_0$

Coefficient of $x = 0$

$\Rightarrow \qquad 3.2\ a_3 - a_1 + 4a_1 = 0 \qquad \Rightarrow \qquad a_3 = -\dfrac{a_1}{2}$

Coefficient of $x^2 = 0$

$\Rightarrow \qquad 4.3.\ a_4 - 2.1\ a_2 - 2a_2 + 4a_2 = 0 \qquad \Rightarrow \qquad a_4 = 0$

Coefficient of $x^3 = 0$

$\Rightarrow \qquad 5.4\ a_5 - 3.2\ a_3 - 3a_3 + 4a_3 = 0$

$$\Rightarrow \qquad a_5 = \frac{a_3}{4} = \frac{1}{4}\left(\frac{-a_1}{2}\right) = -\frac{a_1}{8}$$

$$\Rightarrow \qquad a_5 = -\frac{a_1}{8} \quad \text{and so on.}$$

Substituting these values in assumed solution (1), we get

$$y = a_0 + a_1x - 2a_0x^2 - \frac{a_1}{2}x^3 - \frac{a_1}{8}x^5 + \cdots$$

$$\Rightarrow \qquad y = a_0\,(1 - 2x^2) + a_1x\left(1 - \frac{x^2}{2} - \frac{x^4}{8} - \cdots\right)$$

where a_0 and a_1 are constants.

Example 4. *Find the power series solution of the following differential equation about $x = 0$*

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0.$$

NOTES

Sol. Comparing the given differential equation with the form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \text{ we get}$$

$$P(x) = \frac{-2x}{1-x^2},\ Q(x) = \frac{2}{1-x^2}$$

Since both P(x) and Q(x) are analytic at $x = 0$, hence $x = 0$ is an *ordinary point* of the given equation.

Assume the solution to be

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots \quad \ldots(1)$$

Then, $\quad y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots + n\,a_n\,x^{n-1} + \cdots$

and $\quad y'' = 2.1.\ a_2 + 3.2.\ a_3\,x + 4.3.\ a_4\,x^2 + \cdots + n(n-1)a_n\,x^{n-2} + \cdots$

Substituting these values in given equation, we get

$$(1 - x^2)\,[2.1.\ a_2 + 3.2.\ a_3x + 4.3.\ a_4x^2 + \cdots + n(n-1)\,a_nx^{n-2} + \cdots]$$
$$-2x\,[a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots + n\,a_n\,x^{n-1} + \cdots]$$
$$+ 2\,[a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_n\,x^n + \cdots] = 0$$

Coefficient of $x^0 = 0$

$\Rightarrow \quad 2.1.\ a_2 + 2a_0 = 0 \quad \Rightarrow \quad \boxed{a_2 = -a_0}$

Coefficient of $x = 0$

$\Rightarrow \quad 3.2.\ a_3 - 2a_1 + 2a_1 = 0 \quad \Rightarrow \quad \boxed{a_3 = 0}$

Coefficient of $x^2 = 0$

$\Rightarrow \quad 4.3.a_4 - 2.1.\ a_2 - 4a_2 + 2a_2 = 0$

$\Rightarrow \quad 12a_4 - 4a_2 = 0 \quad \Rightarrow \quad a_4 = \frac{a_2}{3} = -\frac{a_0}{3} \quad \Rightarrow \quad \boxed{a_4 = -\frac{a_0}{3}}$

Coefficient of $x^3 = 0$

$\Rightarrow \quad 5.4.a_5 - 3.2.\ a_3 - 6a_3 + 2a_3 = 0$

$\Rightarrow \quad 20a_5 - 10a_3 = 0 \quad \Rightarrow \quad \boxed{a_5 = 0}$

Coefficient of $x^4 = 0$

$\Rightarrow \quad 6.5.a_6 - 4.3.\ a_4 - 8a_4 + 2a_4 = 0$

$\Rightarrow \quad 30a_6 - 18a_4 = 0 \quad \Rightarrow \quad a_6 = \frac{3}{5}a_4 \quad \Rightarrow \quad \boxed{a_6 = -\frac{a_0}{5}}$

Also, $a_7 = 0,\ a_9 = 0$ and so on.

Substituting these values in assumed solution (1), we get

$$y = a_0 + a_1x - a_0x^2 - \frac{a_0}{3}x^4 - \frac{a_0}{5}x^6 - \cdots\cdots$$

$$\Rightarrow \quad y = a_0\left(1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} - \cdots\right) + a_1x$$

where a_0 and a_1 are constants.

NOTES

Example 5. *Solve in series the differential equation*

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + p(p+1)y = 0.$$

Sol. Here, $\quad P(x) = \dfrac{-2x}{1-x^2}, \quad Q(x) = \dfrac{p(p+1)}{1-x^2}$

Since both P(x) and Q(x) are analytic at $x = 0$ $\therefore$ $x = 0$ is an *ordinary point* of the given differential equation.

Let the solution be $\quad y = a_0 + a_1x + a_2x^2 + \cdots\cdots + a_n x^n + \cdots\cdots = \sum_{n=0}^{\infty} a_n x^n$ ...(1)

$$\therefore \quad \frac{dy}{dx} = \sum_{n=0}^{\infty} n\,a_n\,x^{n-1} \quad ...(2)$$

$$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} n(n-1)\,a_n\,x^{n-2} \quad ...(3)$$

Substituting the above values in the given equation, we get

$$(1-x^2)\sum_{n=0}^{\infty} n(n-1)\,a_n\,x^{n-2} - 2x\sum_{n=0}^{\infty} n\,a_n\,x^{n-1} + p(p+1)\sum_{n=0}^{\infty} a_n\,x^n = 0$$

$$\Rightarrow \quad \sum_{n=0}^{\infty} n(n-1)\,a_n\,x^{n-2} - \sum_{n=0}^{\infty} a_n[n(n-1) + 2n - p(p+1)]x^n = 0$$

$$\Rightarrow \quad \sum_{n=0}^{\infty} n(n-1)\,a_n\,x^{n-2} - \sum_{n=0}^{\infty} a_n(n-p)\,(n+p+1)x^n = 0$$

This is an identity in x.

Coefficient of $x^n = 0$

$$\Rightarrow \quad (n+2)\,(n+1)\,a_{n+2} - (n-p)\,(n+p+1)\,a_n = 0$$

$$\therefore \quad a_{n+2} = \frac{(n-p)(n+p+1)}{(n+2)\,(n+1)}\,a_n$$

Putting $n = 0, 2, 4$, ... etc., we get

$$a_2 = \frac{-p(p+1)}{2.1}\,a_0$$

$$a_4 = \frac{(2-p)(3+p)}{4.3}\,a_2 = \frac{(p-2)(p)(p+1)(p+3)}{4!}\,a_0 \text{ etc.}$$

Again, putting $n = 1, 3, 5$, ... etc., we get

$$a_3 = \frac{(1-p)(p+2)}{3.2}\,a_1 = -\frac{(p-1)\,(p+2)}{3!}\,a_1$$

$$a_5 = \frac{(3-p)(p+4)}{5.4}\,a_3 = \frac{(p-3)(p-1)\,(p+2)(p+4)}{5!}\,a_1 \text{ etc.}$$

Substituting these values in eqn. (1), we get

$$y = a_0\left[1 - \frac{p(p+1)}{2!}x^2 + \frac{(p-2)p(p+1)(p+3)}{4!}x^4 - \ldots\ldots\right]$$

$$+ a_1\left[x - \frac{(p-1)(p+2)}{3!}x^3 + \frac{(p-3)(p-1)(p+2)(p+4)}{5!}x^5 + \ldots\ldots\right]$$

NOTES

Note. Above method is an *aliter* to the method of solution in series discussed before and preferred when, we get the recurrence relation in between a_n and a_{n+2}.

Example 6. *Solve the differential equation* $y'' + (x-1)^2 y' - 4(x-1)y = 0$ *in series about the ordinary point* $x = 1$.

Sol. Put $x = t + 1$ (or $x - 1 = t$)

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt} \qquad \left(\because \frac{dt}{dx} = 1\right)$$

$$\Rightarrow \qquad \frac{d}{dx} \equiv \frac{d}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d^2y}{dt^2}$$

$\therefore$ The given equation becomes,

$$\frac{d^2y}{dt^2} + t^2y' - 4ty = 0 \qquad \ldots(1)$$

Now, $t = 0$ is an *ordinary point.* | given

Assume the solution to be

$$y = a_0 + a_1t + a_2t^2 + a_3t^3 + \ldots + a_nt^n + \ldots \qquad \ldots(2)$$

then $\qquad y' = a_1 + 2a_2t + 3a_3t^2 + \ldots + n\,a_n\,t^{n-1} + \ldots$

and $\qquad y'' = 2a_2 + 3.2.\,a_3t + \ldots + n\,(n-1)\,a_n\,t^{n-2} + \ldots$

Substituting these values in eqn. (1), we get

$$[2a_2 + 3.2.\,a_3t + 4.3.\,a_4\,t^2 + \ldots + n(n-1)\,a_n\,t^{n-2} + \ldots]$$
$$+ t^2\,[a_1 + 2a_2t + 3a_3\,t^2 + 4a_4t^3 + \ldots + n\,a_nt^{n-1} + \ldots]$$
$$- 4t\,[a_0 + a_1t + a_2t^2 + a_3t^3 + \ldots + a_nt^n + \ldots] = 0$$

Coefficient of $t^0 = 0$

$\Rightarrow \qquad 2a_2 = 0 \qquad \Rightarrow \qquad \boxed{a_2 = 0}$

Coefficient of $t = 0$

$\Rightarrow \qquad 3.2.\,a_3 - 4a_0 = 0 \qquad \Rightarrow \qquad \boxed{a_3 = \frac{2a_0}{3}}$

Coefficient of $t^2 = 0$

$\Rightarrow \qquad 4.3.\,a_4 + a_1 - 4a_1 = 0$

$\Rightarrow \qquad 12a_4 = 3a_1 \qquad \Rightarrow \qquad \boxed{a_4 = \frac{a_1}{4}}$

Coefficient of $t^3 = 0$

$\Rightarrow \qquad 5.4.\,a_5 + 2a_2 - 4a_2 = 0 \qquad \Rightarrow \qquad \boxed{a_5 = 0}$

NOTES

Coefficient of $t^4 = 0$

$\Rightarrow$ $6.5.\ a_6 + 3a_3 - 4a_3 = 0$

$$a_6 = \frac{a_3}{6.5} = \frac{2a_0}{6.5.3} \quad \Rightarrow \quad \boxed{a_6 = \frac{a_0}{45}}$$

Now, Coefficient of $t^n = 0$

$\Rightarrow$ $(n + 2)(n + 1)\, a_{n+2} + (n - 1)\, a_{n-1} - 4a_{n-1} = 0$

$$\Rightarrow \quad a_{n+2} = -\frac{(n-5)}{(n+2)(n+1)} a_{n-1}$$

Putting $n = 5, 6, 7, 8, ...$, we get

$$a_7 = 0$$

$$a_8 = \frac{-1}{8.7} a_5 = 0$$

$$a_9 = \frac{-2}{9.8} a_6 = \frac{-2}{9.8} \frac{a_0}{45} = -\frac{a_0}{1620}$$

and so on.

Substituting these values in (2), we get

$$y = a_0 + a_1 t + \frac{2}{3} a_0 t^3 + \frac{a_1}{4} t^4 + \frac{a_0}{45} t^6 - \frac{a_0}{1620} t^9 + \cdots$$

$$= a_0 \left(1 + \frac{2}{3} t^3 + \frac{1}{45} t^6 - \frac{1}{1620} t^9 + \ldots\right) + a_1 \left(t + \frac{t^4}{4}\right)$$

$$\Rightarrow \quad y = a_0 \left[1 + \frac{2}{3}(x-1)^3 + \frac{1}{45}(x-1)^6 - \frac{1}{1620}(x-1)^9 + \ldots\right] + a_1 \left[(x-1) + \frac{(x-1)^4}{4}\right]$$

where a_0 and a_1 are constants.

EXERCISE A

Solve the following equations in series: [*Dashes denote differentiation w.r.t. x*]

1. $\dfrac{d^2y}{dx^2} - y = 0$ 2. $y'' + x^2 y = 0$
3. (*i*) $y'' + xy' + y = 0$ (*ii*) $y'' - xy' + y = 0$
4. (*i*) $y'' - xy' + x^2 y = 0$ (*ii*) $y'' + xy' + x^2 y = 0$
5. $(1 - x^2)\, y'' + 2xy' + y = 0$ 6. $(2 + x^2)\, y'' + xy' + (1 + x)\, y = 0$
7. $(x^2 + 1)\, y'' + xy' - xy = 0$
8. (*i*) $(x^2 - 1)\, y'' + 4xy' + 2y = 0$ (*ii*) $(x^2 - 1)\, y'' + xy' - y = 0$
9. (*i*) $y'' + xy' + (x^2 + 2)\, y = 0$ (*ii*) $(x^2 - 1)y'' + 3xy' + xy = 0$; $y(0) = 4$, $y'(0) = 6$.
10. (*i*) $y'' - xy' + 2y = 0$ near $x = 1$ (*ii*) $y'' + (x - 3)\, y' + y = 0$ near $x = 2$.

Answers

1. $y = a_0 \left(1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \dfrac{x^6}{6!} + \ldots\right) + a_1 \left(x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \ldots\right) = a_0 \cosh x + a_1 \sinh x$

2. $y = a_0 \left(1 - \dfrac{x^4}{3.4} + \dfrac{x^8}{3.4.7.8} - \ldots\right) + a_1 \left(x - \dfrac{x^5}{4.5} + \dfrac{x^9}{4.5.8.9} - \ldots\right)$

NOTES

3. (i) $y = a_0\left(1 - \frac{x^2}{2} + \frac{x^4}{2.4} - \frac{x^6}{2.4.6} + \ldots\right) + a_1\left(x - \frac{x^3}{3} + \frac{x^5}{3.5} - \frac{x^7}{3.5.7} + \ldots\right)$

(ii) $y = a_0\left(1 - \frac{x^2}{2!} - \frac{x^4}{4!} - \frac{3}{6!}x^6 - \frac{3.5}{8!}x^8 + \ldots\right) + a_1 x$

4. (i) $y = a_0\left(1 - \frac{x^4}{12} - \frac{x^6}{90} - \ldots\right) + a_1\left(x + \frac{x^3}{6} - \frac{x^5}{40} - \frac{x^7}{144} + \ldots\right)$

(ii) $y = a_0\left(1 - \frac{x^4}{12} + \frac{x^6}{90} - \ldots\right) + a_1\left(x - \frac{x^3}{6} - \frac{x^5}{40} - \ldots\right)$

5. $y = a_0\left(1 - \frac{x^2}{2} + \frac{x^4}{8} + \ldots\right) + a_1\left(x - \frac{x^3}{2} + \frac{x^5}{40} + \ldots\right)$

6. $y = a_0\left(1 - \frac{x^2}{4} - \frac{x^3}{12} + \frac{5x^4}{96} + \ldots\right) + a_1\left(x - \frac{x^3}{6} - \frac{x^4}{24} + \ldots\right)$

7. $y = a_0\left(1 + \frac{x^3}{6} - \frac{3x^5}{40} + \ldots\right) + a_1\left(x - \frac{x^3}{6} + \frac{x^4}{12} + \frac{3x^5}{40} - \ldots\right)$

8. (i) $y = a_0\,(1 + x^2 + x^4 + \ldots) + a_1\,(x + x^3 + x^5 + \ldots)$

(ii) $y = a_0\left(1 + \frac{x^2}{2} + \frac{x^4}{4} + \ldots\right) + a_1 x$

9. (i) $y = c_0\left(1 - x^2 + \frac{x^4}{4} + \ldots\right) + c_1\left(x - \frac{x^3}{2} + \frac{3}{40}x^5 - \ldots\right)$

(ii) $y = 4 + 6x + \frac{11}{3}x^3 + \frac{1}{2}x^4 + \frac{11}{4}x^5 + \ldots$

10. (i) $y = a_0\left[1 - (x-1)^2 - \frac{1}{3}(x-1)^3 - \ldots\right] + a_1\left[(x-1) + \frac{1}{2}(x-1)^2 - \ldots\right]$

(ii) $y = a_0\left[1 - \frac{1}{2}(x-2)^2 - \frac{1}{6}(x-2)^3 - \frac{1}{12}(x-2)^4 + \ldots\right]$

$+ a_1\left[(x-2) + \frac{1}{2}(x-2)^2 - \frac{1}{6}(x-2)^3 - \frac{1}{6}(x-2)^4 + \ldots\right]$

FROBENIUS METHOD: SERIES SOLUTION WHEN $X = 0$ IS A REGULAR SINGULAR POINT OF THE DIFFERENTIAL EQUATION $\frac{d^2y}{dx^2} + P(x) + \frac{dy}{dx} + Q(x)y = 0$

Steps for solution:

1. Assume $y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \cdots$ ...(1)
2. Substitute from (1) for y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ in given equation.
3. Equate to zero the coefficient of *lowest power* of x. This gives a quadratic equation in m which is known as the *Indicial equation*.
4. Equate to zero, the coefficients of other powers of x to find a_1, a_2, a_3, ... interms of a_0.

5. Substitute the values of a_1, a_2, a_3,...in (1) to get the series solution of the given equation having a_0 as arbitrary constant. Obviously, this is not the complete solution of given equation since the complete solution must have two independent arbitrary constants.

The method of complete solution depends on the nature of roots of the indicial equation.

Case I. When Roots are distinct and do not differ by an integer

e.g., $$m_1 = \frac{1}{2},\ m_2 = 1$$

Let m_1 and m_2 be the roots then complete solution is

$$\boxed{y = c_1\,(y)_{m_1} + c_2\,(y)_{m_2}}$$

SOLVED EXAMPLES

Example 7. *Solve in series the differential equation:*

$$2x\,(1 - x)\,\frac{d^2y}{dx^2} + (5 - 7x)\,\frac{dy}{dx} - 3y = 0.$$

Sol. Comparing the given equation with $\dfrac{d^2y}{dx^2} + \mathrm{P}(x)\dfrac{dy}{dx} + \mathrm{Q}(x)y = 0$, we get

$$\mathrm{P}(x) = \frac{5 - 7x}{2x(1 - x)},\quad \mathrm{Q}(x) = \frac{-3}{2x\,(1 - x)}$$

At $x = 0$, Both P(x) and Q(x) are not analytic, hence $x = 0$ is a *singular point.*

Now, $$x\,\mathrm{P}(x) = \frac{5 - 7x}{2(1 - x)}$$

$$x^2\,\mathrm{Q}(x) = \frac{-3x}{2(1 - x)}$$

At $x = 0$, both x P(x) and x^2 Q(x) are analytic, hence $x = 0$ is a *regular singular* point.

Let us assume

$$y = a_0\,x^m + a_1x^{m+1} + a_2\,x^{m+2} + a_3x^{m+3} + \cdots \quad \ldots(1)$$

Then, $$y' = ma_0x^{m-1} + (m + 1)\,a_1\,x^m + (m + 2)\,a_2\,x^{m+1} + (m + 3)\,a_3\,x^{m+2} + \cdots$$

and $$y' = m(m - 1)\,a_0\,x^{m-2} + (m + 1)\,m\,a_1\,x^{m-1} + (m + 2)\,(m + 1)\,a_2\,x^m + (m + 3)\,(m + 2)\,a_3\,x^{m+1} + \cdots\cdots$$

Substituting these values in given equation, we get

$$2x\,(1 - x)\,[m(m - 1)\,a_0\,x^{m-2} + (m + 1)\,ma_1\,x^{m-1} + (m + 2)\,(m + 1)\,a_2\,x^m + (m + 3)\,(m + 2)\,a_3\,x^{m+1} + \cdots]$$
$$+ (5 - 7x)\,[m\,a_0\,x^{m-1} + (m + 1)\,a_1\,x^m + (m + 2)\,a_2\,x^{m+1} + (m + 3)\,a_3\,x^{m+2} + \cdots]$$
$$- 3\,[a_0\,x^m + a_1\,x^{m+1} + a_2\,x^{m+2} + a_3\,x^{m+3} + \cdots] = 0$$

Now, coefficient of lowest power of $x = 0$

$\Rightarrow$ Coefficient of $x^{m-1} = 0$

$\Rightarrow$ $2m\,(m - 1)\,a_0 + 5m\,a_0 = 0$

$$\Rightarrow \quad (2m^2 + 3m)\, a_0 = 0$$

$$\Rightarrow \quad 2m^2 + 3m = 0 \qquad (\because \; a_0 \neq 0)$$

This is called indicial equation

$$m(2m + 3) = 0$$

$$\Rightarrow \quad \boxed{m = 0, -3/2}$$

Roots are distinct and do not differ by an integer.

Now, Coefficient of $x^m = 0$

$$\Rightarrow \quad 2(m + 1)\, m\, a_1 - 2m\, (m - 1)\, a_0 + 5(m + 1)\, a_1 - 7ma_0 - 3a_0 = 0$$

$$\Rightarrow \quad (m + 1)\,(2m + 5)\, a_1 = (2m^2 - 2m + 7m + 3)\, a_0$$

$$a_1 = \frac{(m+1)(2m+3)}{(m+1)(2m+5)}\, a_0$$

$$\Rightarrow \quad \boxed{a_1 = \frac{2m+3}{2m+5}\, a_0}$$

Coefficient of $x^{m+1} = 0$

$$\Rightarrow \quad 2(m + 2)\,(m + 1)\, a_2 - 2(m + 1)\, m\, a_1 + 5\,(m + 2)\, a_2 - 7(m + 1)\, a_1 - 3a_1 = 0$$

$$\Rightarrow \quad (m + 2)\,(2m + 7)\, a_2 = (2m^2 + 2m + 7m + 7 + 3)\, a_1$$

$$= (2m^2 + 9m + 10)\, a_1 = (2m + 5)\,(m + 2)\, a_1$$

$$\Rightarrow \quad a_2 = \frac{2m+5}{2m+7}\, a_1 = \frac{2m+5}{2m+7} \cdot \frac{2m+3}{2m+5}\, a_0$$

$$\Rightarrow \quad \boxed{a_2 = \frac{2m+3}{2m+7}\, a_0}$$

Similarly, $$a_3 = \frac{2m+7}{2m+9}\, a_2 = \frac{2m+7}{2m+9} \cdot \frac{2m+3}{2m+7}\, a_0$$

$$\Rightarrow \quad \boxed{a_3 = \frac{2m+3}{2m+9}\, a_0}$$

and so on.

Hence, from (1),

$$y = x^m \left[a_0 + \frac{2m+3}{2m+5}\, a_0\, x + \frac{2m+3}{2m+7}\, a_0\, x^2 + \frac{2m+3}{2m+9}\, a_0\, x^3 + \ldots \right]$$

$$\Rightarrow \quad y = a_0\, x^m \left[1 + \left(\frac{2m+3}{2m+5} \right) x + \left(\frac{2m+3}{2m+7} \right) x^2 + \left(\frac{2m+3}{2m+9} \right) x^3 + \ldots \right] \qquad \ldots(2)$$

Now, $$y_1 = (y)_{m=0}$$

$$\boxed{y_1 = a_0 \left[1 + \frac{3}{5}\, x + \frac{3}{7}\, x^2 + \frac{3}{9}\, x^3 + \ldots \right]} \qquad \ldots(3)$$

Also, $$y_2 = (y)_{m=-3/2} = a_0\, x^{-3/2}\, (1 + 0\, .\, x + 0\, .\, x^2 + 0\, .\, x^3 + \ldots)$$

$$y_2 = a_0\, x^{-3/2} \qquad \ldots(4)$$

Hence the complete solution is given by

$$y = c_1\, y_1 + c_2\, y_2 = c_1\, a_0 \left(1 + \frac{3}{5}\, x + \frac{3}{7}\, x^2 + \frac{3}{9}\, x^3 + \ldots \right) + c_2 a_0\, x^{-3/2}$$

NOTES

$$\Rightarrow \qquad y = A\left(1 + \frac{3}{5}x + \frac{3}{7}x^2 + \frac{3}{9}x^3 + \ldots\right) + Bx^{-3/2}$$

where A and B are constants.

NOTES

Example 8. *Solve in series the differential equation*

$$2x^2 \frac{d^2y}{dx^2} + (2x^2 - x)\frac{dy}{dx} + y = 0.$$

Sol. Comparing the given equation with $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0$, we get

$$P(x) = \frac{2x^2 - x}{2x^2} = 1 - \frac{1}{2x} \quad \text{and} \quad Q(x) = \frac{1}{2x^2}$$

At $x = 0$, Both P(x) and Q(x) are not analytic, hence $x = 0$ is a *singular point.*

Now, $\qquad x\,P(x) = x - \frac{1}{2} \quad$ and $\quad x^2\,Q(x) = \frac{1}{2}$

Since both x P(x) and x^2 Q(x) are analytic at $x = 0$, hence $x = 0$ is a *regular singular point.*

Let us assume

$$y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + \ldots \qquad \ldots(1)$$

Then, $\quad y' = m\,a_0 x^{m-1} + (m+1)\,a_1 x^m + (m+2)\,a_2 x^{m+1} + (m+3)\,a_3 x^{m+2} + \ldots$

and $\quad y'' = m(m-1)\,a_0 x^{m-2} + (m+1)\,m\,a_1 x^{m-1} + (m+2)(m+1)\,a_2 x^m$
$\qquad + (m+3)(m+2)\,a_3 x^{m+1} + \ldots$

Substituting these values in given equation, we get

$$2x^2 [m(m-1)\,a_0 x^{m-2} + (m+1)m\,a_1 x^{m-1} + (m+2)(m+1)\,a_2 x^m$$
$$+ (m+3)(m+2)\,a_3 x^{m+1} + \ldots] + (2x^2 - x)[m\,a_0 x^{m-1} + (m+1)\,a_1 x^m$$
$$+ (m+2)\,a_2 x^{m+1} + (m+3)\,a_3 x^{m+2} + \ldots]$$
$$+ [a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + \ldots] = 0$$

Now, Coeff. of lowest power of $x = 0$ *i.e.,* Coeff. of $x^m = 0$

$$2m(m-1)\,a_0 - m\,a_0 + a_0 = 0$$

$$\Rightarrow \qquad (2m^2 - 3m + 1)\,a_0 = 0$$

$$\Rightarrow \qquad (2m-1)(m-1) = 0 \text{ (since } a_0 \neq 0)$$

which is indicial equation.

Its roots are $\qquad \boxed{m = 1, \frac{1}{2}}$

Roots are distinct and donot differ by an integer.

Now, $\qquad$ Coefficient of $x^{m+1} = 0$

$$\Rightarrow \qquad 2m(m+1)\,a_1 + 2m\,a_0 - (m+1)\,a_1 + a_1 = 0$$

$$\Rightarrow \qquad (2m^2 + m)\,a_1 + 2m\,a_0 = 0$$

$$\Rightarrow \qquad \boxed{a_1 = -\frac{2}{2m+1}a_0} \qquad | \because \; m \neq 0$$

Coefficient of $x^{m+2} = 0$

$$\Rightarrow \qquad 2(m+2)(m+1)\,a_2 + 2(m+1)\,a_1 - (m+2)\,a_2 + a_2 = 0$$

$$\Rightarrow \qquad (2m^2 + 5m + 3)\,a_2 + 2(m+1)\,a_1 = 0$$

$$\Rightarrow \qquad (2m+3)(m+1)\,a_2 + 2(m+1)\,a_1 = 0$$

NOTES

$$\Rightarrow \qquad a_2 = \frac{-2}{2m+3} a_1 = \frac{(-2)}{2m+3} \cdot \frac{(-2)}{2m+2} a_0$$

$$\Rightarrow \qquad \boxed{a_2 = \frac{4}{(2m+1)(2m+3)} a_0}$$

Similarly, we can find

$$a_3 = \frac{-8}{(2m+1)(2m+3)(2m+5)} a_0$$

$$a_4 = \frac{16}{(2m+1)(2m+3)(2m+5)(2m+7)} a_0$$

and so on.

$$\therefore \qquad y = a_0 x^m \left[1 - \frac{2}{2m+1} x + \frac{4}{(2m+1)(2m+3)} x^2 - \frac{8}{(2m+1)(2m+3)(2m+5)} x^3 + \ldots\right] \qquad \ldots(2)$$

Now, $y_1 = (y)_{m=1}$

$$y_1 = a_0 x \left[1 - \frac{2}{3} x + \frac{4}{3.5} x^2 - \frac{8}{3.5.7} x^3 + \ldots\right]$$

or

$$\boxed{y_1 = a_0 x \left(1 - \frac{2}{3} x + \frac{2^2}{3.5} x^2 - \frac{2^3}{3.5.7} x^3 + \ldots\right)} \qquad \ldots(3)$$

and $y_2 = (y)_{m=1/2}$

$$\boxed{y_2 = a_0 x^{1/2} \left[1 - x + \frac{1}{2} x^2 - \frac{1}{6} x^3 + \ldots\right]} \qquad \ldots(4)$$

Hence the complete solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 a_0 x \left(1 - \frac{2}{3} x + \frac{2^2}{3.5} x^2 - \frac{2^3}{3.5.7} x^3 + \ldots\right) + c_2 a_0 \sqrt{x} \left(1 - x + \frac{1}{2} x^2 - \frac{1}{6} x^3 + \ldots\right)$$

$$\Rightarrow \qquad \boxed{y = Ax \left(1 - \frac{2}{3} x + \frac{2^2}{3.5} x^2 - \frac{2^3}{3.5.7} x^3 + \ldots\right) + B\sqrt{x} \left(1 - x + \frac{1}{2} x^2 - \frac{1}{6} x^3 + \ldots\right)}$$

where A and B are constants.

EXERCISE B

Solve in series:

1. $9x(1-x) \dfrac{d^2y}{dx^2} - 12 \dfrac{dy}{dx} + 4y = 0$

2. $x(2+x^2) \dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 6xy = 0$

3. $3x \dfrac{d^2y}{dx^2} + 2 \dfrac{dy}{dx} + y = 0$

4. $2x^2 \dfrac{d^2y}{dx^2} - x \dfrac{dy}{dx} + (1-x^2) y = 0$

NOTES

5. $2x^2 y'' + xy' - (x+1)y = 0$

6. $2x(1-x)\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + 3y = 0$

7. $2x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + (x-5)y = 0$

8. $y'' + \frac{1}{4x}y' + \frac{1}{8x^2}y = 0$

9. $2x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + (x^2+1)y = 0$

10. $4x\frac{d^2y}{dx^2} + 2(1-x)\frac{dy}{dx} - y = 0$.

11. $2x^2y'' + 7x(x+1)y' - 3y = 0$

12. $2x^2y'' + x(2x+1)y' - y = 0$

Answers

1. $y = A\left(1+\frac{1}{3}x+\frac{1.4}{3.6}x^2+\frac{1.4.7}{3.6.9}x^3+\ldots\right) + Bx^{7/3}\left(1+\frac{8}{10}x+\frac{8.11}{10.13}x^2+\frac{8.11.14}{10.13.16}x^3+\ldots\right)$

2. $y = A\left(1+3x^2+\frac{3}{5}x^4-\frac{1}{15}x^6+\ldots\right) + Bx^{3/2}\left(1+\frac{3}{8}x^2-\frac{3.1}{8.16}x^4+\frac{5.3.1}{8.16.24}x^6-\ldots\right)$

3. $y = A\left(1-\frac{x}{2}+\frac{x^2}{20}-\frac{x^3}{480}+\ldots\right) + Bx^{1/3}\left(1-\frac{x}{4}+\frac{x^2}{56}-\frac{x^3}{1680}+\ldots\right)$

4. $y = Ax\left(1+\frac{x^2}{2.5}+\frac{x^4}{2.4.5.9}+\ldots\right) + Bx^{1/2}\left(1+\frac{x^2}{2.3}+\frac{x^4}{2.4.3.7}+\ldots\right)$

5. $y = Ax\left(1+\frac{1}{5}x+\frac{1}{70}x^2+\ldots\right) + Bx^{-1/2}\left(1-x-\frac{1}{2}x^2+\ldots\right)$

6. $y = A\left(1-3x+\frac{3x^2}{1.3}+\frac{3x^3}{3.5}+\frac{3x^4}{5.7}+\ldots\right) + B\sqrt{x}\,(1-x)$

7. $y = c_1x^{5/2}\left(1-\frac{x}{9}+\frac{x^2}{198}-\frac{x^3}{7722}+\ldots\right) + c_2x^{-1}\left(1+\frac{x}{5}+\frac{x^2}{30}+\frac{x^3}{90}+\ldots\right)$

8. $y = A\sqrt{x} + Bx^{1/4}$

9. $y = Ax\left(1-\frac{x^2}{10}+\frac{x^4}{360}-\ldots\right) + Bx^{1/2}\left(1-\frac{x^2}{6}+\frac{x^4}{168}-\ldots\right)$

10. $y = A\left(1+\frac{x}{2.1!}+\frac{x^2}{2^2.2!}+\frac{x^3}{2^3.3!}+\ldots\right) + B\sqrt{x}\left(1+\frac{x}{1.3}+\frac{x^2}{1.3.5}+\frac{x^3}{1.3.5.7}+\ldots\right)$.

11. $y = A\sqrt{x}\left(1-\frac{7}{18}x+\frac{49}{264}x^2\ldots\right) + Bx^{-3}\left(1-\frac{21}{5}x+\frac{49}{5}x^2\ldots\right)$

12. $y = Ax^{-1/2}\left(1-x+\frac{x^2}{2}-\frac{x^3}{6}+\ldots.\right) + Bx\left(1-\frac{2x}{5}+\frac{4x^2}{35}-\frac{8x^3}{315}+\ldots.\right)$

Case II. When Roots are Equal *e.g.*, $m_1 = m_2 = 0$

Complete solution is

$$y = c_1\,(y)_{m_1} + c_2\left(\frac{\partial y}{\partial m}\right)_{m_1}$$

SOLVED EXAMPLES

NOTES

Example 9. *Solve in series:* $x(x-1)\dfrac{d^2y}{dx^2} + (3x-1)\dfrac{dy}{dx} + y = 0$.

Sol. Comparing the given equation with

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0, \text{ we get}$$

$$P(x) = \frac{3x-1}{x(x-1)} \text{ and } Q(x) = \frac{1}{x(x-1)}$$

At $x = 0$, Both $P(x)$ and $Q(x)$ are not analytic, hence $x = 0$ is a *singular point.*

Now, $\quad x\,P(x) = \dfrac{3x-1}{x-1}$ and $x^2\,Q(x) = \dfrac{x}{x-1}$

Both $x\,P(x)$ and $x^2\,Q(x)$ are analytic at $x = 0$, hence $x = 0$ is a *regular singular point.*

Let us assume

$$y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + \ldots \qquad \ldots(1)$$

Then, $\quad y' = m\,a_0\,x^{m-1} + (m+1)\,a_1\,x^m + (m+2)\,a_2\,x^{m+1} + (m+3)\,a_3\,x^{m+2} + \ldots$

and $\quad y'' = m(m-1)\,a_0\,x^{m-2} + (m+1)m\,a_1\,x^{m-1} + (m+2)(m+1)\,a_2\,x^m + (m+3)(m+2)\,a_3\,x^{m+1} + \ldots$

Substituting these values in given equation, we get

$$x(x-1)[m(m-1)\,a_0\,x^{m-2} + (m+1)\,m\,a_1\,x^{m-1} + (m+2)(m+1)\,a_0\,x^m + (m+3)(m+2)\,a_3\,x^{m+1} + \ldots]$$
$$+ (3x-1)[m\,a_0\,x^{m-1} + (m+1)\,a_1\,x^m + (m+2)\,a_2\,x^{m+1} + (m+3)\,a_3\,x^{m+2} + \ldots]$$
$$+ [a_0\,x^m + a_1\,x^{m+1} + a_2\,x^{m+2} + a_3\,x^{m+3} + \ldots] = 0$$

Now, Coefficient of lowest power of $x = 0$

$\Rightarrow$ Coefficient of $x^{m-1} = 0$

$\Rightarrow$ $-m(m-1)\,a_0 - m\,a_0 = 0 \;\Rightarrow\; -m^2\,a_0 = 0$

$m^2 = 0 \qquad (\because\; a_0 \neq 0)$

which is *Indicial equation*

Its roots are $\boxed{m = 0, 0}$

Roots are equal.

Now, Coefficient of $x^m = 0$

$\Rightarrow$ $m(m-1)\,a_0 - (m+1)\,m\,a_1 + 3m\,a_0 - (m+1)\,a_1 + a_0 = 0$

$\Rightarrow$ $(m+1)^2 a_0 - (m+1)^2\,a_1 = 0$

$\Rightarrow$ $\boxed{a_1 = a_0} \qquad (\because\; m \neq -1)$

Coefficient of $x^{m+1} = 0$

$\Rightarrow$ $(m+1)\,m\,a_1 - (m+2)(m+1)\,a_2 + 3(m+1)\,a_1 - (m+2)a_2 + a_1 = 0$

$\Rightarrow$ $(m+2)^2\,a_1 - (m+2)^2\,a_2 = 0$

$\Rightarrow$ $a_2 = a_1 \qquad (\because\; m \neq -2)$

$\Rightarrow$ $\boxed{a_2 = a_0}$

Similarly, we can show that

$a_3 = a_0$

$a_4 = a_0$ and so on.

$\therefore \quad y = a_0 x^m (1 + x + x^2 + x^3 + ...)$ | From (1)

Now, $y_1 = (y)_{m=0} = a_0 x^0 (1 + x + x^2 + x^3 + ...) = a_0 (1 + x + x^2 + x^3 + ...)$

$$y_2 = \left(\frac{\partial y}{\partial m}\right)_{m=0} = [a_0 (1 + x + x^2 + x^3 + ...) x^m \log x]_{m=0}$$

$$= a_0 \log x(1 + x + x^2 + x^3 + ...)$$

Hence the complete solution is given by

$$y = c_1 y_1 + c_2 y_2 = c_1 a_0 (1 + x + x^2 + x^3 + ...) + c_2 a_0 \log x(1 + x + x^2 + x^3 + ...)$$

$$y = (\text{A} + \text{B} \log x)(1 + x + x^2 + x^3 + ...)$$

where A and B are constants.

Example 10. *Solve in series the differential equation:* $x \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - y = 0$.

Sol. Comparing with the equation $\dfrac{d^2y}{dx^2} + \text{P}(x)\dfrac{dy}{dx} + \text{Q}(x)\, y = 0$, we get

$$\text{P}(x) = \frac{1}{x} \text{ and } \text{Q}(x) = -\frac{1}{x}$$

Since at $x = 0$, both P(x) and Q(x) are not analytic $\therefore$ $x = 0$ is a *singular point.*

Also, $x\,\text{P}(x) = 1$ and $x^2\,\text{Q}(x) = -x$

Both $x\,\text{P}(x)$ and $x^2\,\text{Q}(x)$ are analytic at $x = 0$ $\therefore$ $x = 0$ is a *regular singular point.*

Let us assume

$$y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + ... \quad ...(1)$$

Then, $y' = m a_0 x^{m-1} + (m + 1) a_1 x^m + (m + 2) a_2 x^{m+1} + (m + 3) a_3 x^{m+2} + ...$

and $y'' = m(m - 1) a_0 x^{m-2} + (m + 1) m a_1 x^{m-1} + (m + 2)(m + 1) a_2 x^m + (m + 3)(m + 2) a_3 x^{m+1} + ...$

Substituting these values in the given equation, we get

$$x [m (m - 1) a_0 x^{m-2} + (m + 1)m a_1 x^{m-1} + (m + 2)(m + 1) a_2 x^m + (m + 3)(m + 2) a_3 x^{m+1} + ...]$$
$$+ [m a_0 x^{m-1} + (m + 1) a_1 x^m + (m + 2) a_2 x^{m+1} + (m + 3) a_3 x^{m+2} + ...]$$
$$- [a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + ...] = 0$$

Now, Coefficient of $x^{m-1} = 0$

$\Rightarrow \quad m(m - 1) a_0 + m a_0 = 0$

$\Rightarrow \quad m^2 a_0 = 0 \Rightarrow m^2 = 0 \qquad (\because \; a_0 \neq 0)$

which is Indicial equation.

Its roots are $\boxed{m = 0, 0}$ *which are equal.*

Coefficient of $x^m = 0$

$\Rightarrow \quad (m + 1) m a_1 + (m + 1) a_1 - a_0 = 0 \quad \Rightarrow \quad (m + 1)^2 a_1 = a_0$

$$\Rightarrow \quad \boxed{a_1 = \frac{a_0}{(m + 1)^2}}$$

Coefficient of $x^{m+1} = 0$

$\Rightarrow \quad (m + 2)(m + 1) a_2 + (m + 2) a_2 - a_1 = 0 \quad \Rightarrow \quad (m + 2)^2 a_2 = a_1$

$\Rightarrow \qquad a_2 = \dfrac{a_1}{(m+2)^2} \quad \Rightarrow \quad \boxed{a_2 = \dfrac{a_0}{(m+1)^2\,(m+2)^2}}$

Similarly, $\qquad a_3 = \dfrac{a_0}{(m+1)^2\,(m+2)^2\,(m+3)^2}$ and so on.

NOTES

$\therefore$ From (1),

$$y = a_0 x^m \left[1 + \frac{x}{(m+1)^2} + \frac{x^2}{(m+1)^2\,(m+2)^2} + \frac{x^3}{(m+1)^2\,(m+2)^2\,(m+3)^2} + \ldots\right] \quad \ldots(2)$$

Now, $\qquad y_1 = (y)_{m=0} = a_0 \left[1 + x + \dfrac{x^2}{(2\,!)^2} + \dfrac{x^3}{(3\,!)^2} + \ldots\right] \qquad \ldots(3)$

To get the second independent solution, differentiate (1) partially w.r.t. m.

$$\frac{\partial y}{\partial m} = a_0 x^m \log x \left[1 + \frac{x}{(m+1)^2} + \frac{x^2}{(m+1)^2\,(m+2)^2} + \frac{x^3}{(m+1)^2\,(m+2)^2\,(m+3)^2} + \ldots\right]$$

$$+ a_0 x^m \left[-\frac{2x}{(m+1)^3} - \frac{2}{(m+1)^2\,(m+2)^2}\left\{\frac{1}{m+1} + \frac{1}{m+2}\right\} x^2 - \frac{2}{(m+1)^2\,(m+2)^2\,(m+3)^2}\left\{\frac{1}{m+1} + \frac{1}{m+2} + \frac{1}{m+3}\right\} x^3 - \ldots\right]$$

The second solution is $y_2 = \left(\dfrac{\partial y}{\partial m}\right)_{m=0} = a_0 \log x \left[1 + x + \dfrac{x^2}{(2\,!)^2} + \dfrac{x^3}{(3\,!)^2} + \ldots\right]$

$$- 2a_0 \left[x + \frac{1}{(2\,!)^2}\left(1 + \frac{1}{2}\right) x^2 + \frac{1}{(3\,!)^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right) x^3 + \ldots\right]$$

$$= y_1 \log x - 2a_0 \left[x + \frac{1}{(2\,!)^2} + \left(1 + \frac{1}{2}\right) x^2 + \frac{1}{(3\,!)^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right) x^3 + \ldots\right]$$

Hence the complete solution is

$$y = c_1 y_1 + c_2 y_2 = (c_1 a_0 + c_2 a_0 \log x) \left[1 + x + \frac{x^2}{(2\,!)^2} + \frac{x^3}{(3\,!)^2} + \ldots\right]$$

$$- 2c_2 a_0 \left[x + \frac{1}{(2\,!)^2}\left(1 + \frac{1}{2}\right) x^2 + \frac{1}{(3\,!)^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right) x^3 + \ldots\right]$$

$\Rightarrow \qquad y = (\mathrm{A} + \mathrm{B} \log x) \left[1 + x + \dfrac{x^2}{(2\,!)^2} + \dfrac{x^3}{(3\,!)^2} + \ldots\right]$

$$- 2\mathrm{B} \left[x + \frac{1}{(2\,!)^2}\left(1 + \frac{1}{2}\right) x^2 + \frac{1}{(3\,!)^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right) x^3 + \ldots\right]$$

where $c_1 a_0 = \mathrm{A}$, $c_2 a_0 = \mathrm{B}$.

EXERCISE C

Solve in series:

1. (*i*) $xy'' + (1 + x)\,y' + 2y = 0$ (*ii*) $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - xy = 0$

2. $x^2 \dfrac{d^2y}{dx^2} + x(x-1)\dfrac{dy}{dx} + (1-x)y = 0$ **3.** $(x - x^2)\dfrac{d^2y}{dx^2} + (1-5x)\dfrac{dy}{dx} - 4y = 0$

4. $(x - x^2)y'' + (1 - x)y' - y = 0$ **5.** $x^2 y'' - x(1 + x)y' + y = 0$

NOTES

6. $xy'' + y' + x^2y = 0$

7. $xy'' + y' + xy = 0.$ (*Bessel's equation of order zero*)

Answers

1. (*i*) $y = A\left(1 - 2x + \dfrac{3}{2!}x^2 - \dfrac{4}{3!}x^3 + \ldots\right) + B\left[y_1 \log x + a_0\left(3x - \dfrac{13}{4}x^2 + \ldots\right)\right]$

(*ii*) $y = (A + B\log x)\left(1 + \dfrac{x^2}{2^2} + \dfrac{x^4}{2^2 . 4^2} + \ldots\right) - B\left(\dfrac{x^2}{2^2} + \dfrac{3x^4}{2 . 4^3} + \ldots\right)$

2. $y = Ax + B\left[x\log x - x + \dfrac{x^2}{4} - \ldots\right]$

3. $y = A(1^2 + 2^2x + 3^2x^2 + 4^2x^3 + \ldots) + B[y_1 \log x - 2a_0 (1.2x + 2.3x^2 + 3.4x^3 + \ldots)]$

4. $y = A\left(1 + x + \dfrac{2}{4}x^2 + \dfrac{2.5}{4.9}x^3 + \ldots\right) + B\left[y_1 \log x + a_0\left(-2x - x^2 - \dfrac{14}{27}x^3 - \ldots\right)\right]$

5. $y = Ax\left(1 + x + \dfrac{1}{2}x^2 + \dfrac{1}{2.3}x^3 + \ldots\right) + B\left[y_1 \log x + a_0 x^2\left(-1 - \dfrac{3}{4}x + \ldots\right)\right]$

6. $y = A\left[1 - \dfrac{x^3}{3^2} + \dfrac{x^6}{3^4(2!)^2} - \dfrac{x^9}{3^6(3!)^2} + \ldots\right]$

$$+ B\left[y_1 \log x + 2a_0\left\{\frac{x^3}{3^3} - \frac{1}{3^5(2!)^2}\left(1 + \frac{1}{2}\right)x^6 + \ldots\right\}\right]$$

7. $y = A\left(1 - \dfrac{x^2}{2^2} + \dfrac{x^4}{2^2 . 4^2} - \dfrac{x^6}{2^2 . 4^2 . 6^2} + \ldots\right) + B\left[y_1 \log x + a_0\left\{\dfrac{x^2}{2^2} - \dfrac{1}{2^2 . 4^2}\left(1 + \dfrac{1}{2}\right)x^4\right.\right.$

$$\left.\left. + \frac{1}{2^2 . 4^2 . 6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right)x^6 - \ldots\right\}\right].$$

Case III. When Roots are Distinct, Differ by Integer and Making a Coefficient of y Infinite

Let m_1 and m_2 be the roots such that $m_1 > m_2$.

In this case, if some of the coefficients of y become infinite when $m = m_2$, we modify the form of y by replacing a_0 by $b_0 (m - m_2)$.

Complete solution is

$$y = c_1 (y)_{m_1} + c_2 \left(\frac{\partial y}{\partial m}\right)_{m_2}.$$

Remark. We can also obtain two independent solutions by putting $m = m_2$ (value of m for which some coefficients of y become infinite) in modified form of y and $\dfrac{\partial y}{\partial m}$. The result of putting $m = m_1$ in y will give a numerical multiple of that obtained by putting $m = m_2$.

SOLVED EXAMPLES

NOTES

Example 11. *Obtain the series solution of the Bessel's equation of order two*

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - 4)\, y = 0 \quad \textit{near } x = 0.$$

Sol. Comparing the given equation with the form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \text{ we get}$$

$$P(x) = \frac{1}{x} \quad \text{and} \quad Q(x) = \frac{x^2 - 4}{x^2} = 1 - \frac{4}{x^2}$$

At $x = 0$, both $P(x)$ and $Q(x)$ are not analytic.

Therefore $x = 0$ is a *singular point.*

Also, $x\,P(x) = 1$ and $x^2\,Q(x) = x^2 - 4$

Both $x\,P(x)$ and $x^2\,Q(x)$ are analytic at $x = 0$

$\therefore$ $x = 0$ is a *regular singular point.*

Let us assume,

$$y = a_0x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + \ldots \qquad \ldots(1)$$

Then, $$\frac{dy}{dx} = m\, a_0 x^{m-1} + (m + 1)\, a_1 x^m + (m + 2)\, a_2 x^{m+1} + (m + 3)\, a_3 x^{m+2} + \ldots$$

and $$\frac{d^2y}{dx^2} = m(m - 1)\, a_0\, x^{m-2} + (m + 1)\, ma_1\, x^{m-1} + (m + 2)\,(m + 1)\, a_2\, x^m + (m + 3)\,(m + 2)\, a_3x^{m+1} + \ldots$$

Substituting these values in the given equation, we get

$$x^2\, [m(m - 1)\, a_0\, x^{m-2} + (m + 1)\, ma_1\, x^{m-1} + (m + 2)\,(m + 1)\, a_2\, x^m + (m + 3)\,(m + 2)\, a_3\, x^{m+1} + \ldots]$$
$$+ x\, [m\, a_0\, x^{m-1} + (m + 1)\, a_1\, x^m + (m + 2)\, a_2\, x^{m+1} + (m + 3)\, a_3\, x^{m+2} + \ldots]$$
$$+ (x^2 - 4)\, [a_0\, x^m + a_1\, x^{m+1} + a_2\, x^{m+2} + a_3\, x^{m+3} + \ldots] = 0$$

Now, Coefficient of lowest power of $x = 0$

$\Rightarrow$ Coefficient of $x^m = 0$

$\Rightarrow$ $m(m - 1)a_0 + m\, a_0 - 4a_0 = 0$ $\Rightarrow$ $(m^2 - 4)\, a_0 = 0$

$\Rightarrow$ $m^2 - 4 = 0$ (*Indicial equation*) $\mid \because\ a_0 \neq 0$

$$\boxed{m = -2,\ 2}$$

Roots are distinct and differ by integer.

Now, Coefficient of $x^{m+1} = 0$

$(m + 1)m\, a_1 + (m + 1)\, a_1 - 4a_1 = 0$

$\Rightarrow$ $(m^2 + 2m - 3)\, a_1 = 0$ $\Rightarrow$ $(m + 3)\,(m - 1)\, a_1 = 0$

$\Rightarrow$ $\boxed{a_1 = 0}$ | Since $m \neq 1$, and $m \neq -3$

Coefficient of $x^{m+2} = 0$

$\Rightarrow (m + 2)\,(m + 1)\, a_2 + (m + 2)\, a_2 + a_0 - 4a_2 = 0$

$\Rightarrow$ $(m^2 + 4m)\, a_2 + a_0 = 0$

NOTES

$$\Rightarrow \qquad \boxed{a_2 = \frac{-a_0}{m(m+4)}}$$

Coefficient of $x^{m+3} = 0$

$\Rightarrow (m + 3)(m + 2) a_3 + (m + 3) a_3 + a_1 - 4a_3 = 0$

$\Rightarrow \qquad (m + 1)(m + 5) a_3 = - a_1$

$$\Rightarrow \qquad \boxed{a_3 = 0} \qquad \Big|\because \ a_1 = 0$$

Also, coefficient of $x^{m+4} = 0$

$(m + 2)(m + 6) a_4 + a_2 = 0$

$$\Rightarrow \qquad a_4 = \frac{-a_2}{(m+2)(m+6)} = \frac{a_0}{m(m+2)(m+4)(m+6)}$$

$$\therefore \qquad \boxed{a_4 = \frac{a_0}{m(m+2)(m+4)(m+6)}}$$

Similarly, $a_5 = a_7 = a_9 = \ldots = 0$

$$a_6 = \frac{-a_0}{m(m+2)(m+4)^2(m+6)(m+8)} \text{ etc.}$$

Substituting above obtained values in assumed y given by eqn. (1), we get

$$y = a_0 x^m \left[1 - \frac{x^2}{m(m+4)} + \frac{x^4}{m(m+2)(m+4)(m+6)} - \frac{x^6}{m(m+2)(m+4)^2(m+6)(m+8)} + \ldots\right] \quad \ldots(2)$$

Putting $m = 2$ (the greater of the two roots) in (2), the first solution is

$$y_1 = a_0 x^2 \left(1 - \frac{x^2}{2.6} + \frac{x^4}{2.4.6.8} - \frac{x^6}{2.4.6^2.8.10} + \ldots\right)$$

If we put $m = -2$ in (1), the coefficients become infinite due to the presence of the factor $(m + 2)$ in the denominator. To overcome this difficulty, let $a_0 = b_0(m + 2)$ so that

$$y = b_0 x^m \left[(m+2) - \frac{(m+2)x^2}{m(m+4)} + \frac{x^4}{m(m+4)(m+6)} - \frac{x^6}{m(m+4)^2(m+6)(m+8)} + \ldots\right]$$

Differentiating partially w.r.t. m, we get

$$\frac{\partial y}{\partial m} = b_0 x^m \log x \left[(m+2) - \frac{(m+2)x^2}{m(m+4)} + \frac{x^4}{m(m+4)(m+6)} - \ldots\right]$$
$$+ b_0 x^m \left[1 - \frac{(m+2)}{m(m+4)}\left\{\frac{1}{m+2} - \frac{1}{m} - \frac{1}{m+4}\right\} x^2 + \frac{1}{m(m+4)(m+6)}\left\{-\frac{1}{m} - \frac{1}{m+4} - \frac{1}{m+6}\right\} x^4 \ldots\right]$$

The second solution is $y_2 = \left(\dfrac{\partial y}{\partial m}\right)_{m=-2}$

$$= b_0 x^{-2} \log x \left[\frac{x^4}{(-2)(2)(4)} - \frac{x^6}{(-2)(2)^2(4)(6)} \ldots \right]$$

$$+ b_0 x^{-2} \left[1 - \frac{x^2}{(-2)(2)} + \frac{1}{(-2)(2)(4)} \left(\frac{1}{2} - \frac{1}{2} - \frac{1}{4} \right) x^4 \ldots \right]$$

NOTES

$$= b_0 x^2 \log x \left[-\frac{1}{2^2.4} + \frac{x^2}{2^3.4.6} \ldots \right] + b_0 x^{-2} \left[1 + \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} + \ldots \right]$$

Hence the complete solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= \mathrm{A}x^2 \left[\left(1 - \frac{x^2}{2.6} + \frac{x^4}{2.4.6.8} - \frac{x^6}{2.4.6^2.8.10} + \ldots \right) \right] + \mathrm{B} \left[x^2 \log x \left(-\frac{1}{2^2.4} + \frac{x^2}{2^3.4.6} \ldots \right) \right.$$

$$\left. + x^{-2} \left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} + \ldots \right) \right]$$

where $\mathrm{A} = c_1 a_0$, $\mathrm{B} = c_2 b_0$.

Example 12. *Solve in series the differential equation* $x^2 \dfrac{d^2y}{dx^2} + 5x \dfrac{dy}{dx} + x^2 y = 0$.

Sol. Comparing the given equation with the form

$$\frac{d^2y}{dx^2} + \mathrm{P}(x) \frac{dy}{dx} + \mathrm{Q}(x)\, y = 0, \text{ we get}$$

$$\mathrm{P}(x) = \frac{5}{x}, \quad \mathrm{Q}(x) = 1$$

At $x = 0$, since $\mathrm{P}(x)$ is not analytic $\therefore$ $x = 0$ is a *singular point.*

Also, $x\,\mathrm{P}(x) = 5$

$x^2\,\mathrm{Q}(x) = 0$

Since both $x\,\mathrm{P}(x)$ and $x^2\,\mathrm{Q}(x)$ are analytic at $x = 0$ $\therefore$ $x = 0$ is a *regular singular point.*

Let us assume

$$y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + \ldots \qquad \ldots(1)$$

$$\therefore \quad \frac{dy}{dx} = m\, a_0\, x^{m-1} + (m+1)\, a_1\, x^m + (m+2)\, a_2\, x^{m+1} + \ldots \qquad \ldots(2)$$

and

$$\frac{d^2y}{dx^2} = m\,(m-1)\, a_0\, x^{m-2} + (m+1)\, m\, a_1\, x^{m-1} + (m+2)\,(m+1)\, a_2\, x^m + \ldots \qquad \ldots(3)$$

Substituting the above values in given equation, we get

$$x^2\, [m(m-1)\, a_0\, x^{m-2} + (m+1)\, m a_1\, x^{m-1} + (m+2)\,(m+1)\, a_2\, x^m + \ldots]$$
$$+ 5x\, [m a_0\, x^{m-1} + (m+1)\, a_1\, x^m + (m+2)\, a_2\, x^{m+1} + \ldots]$$
$$+ x^2\, [a_0 x^m + a_1\, x^{m+1} + a_2\, x^{m+2} + \ldots] = 0 \qquad \ldots(4)$$

Equating the coefficient of lowest power of x to zero, we get

$$m(m-1)\, a_0 + 5m a_0 = 0 \qquad [\text{Coeff. of } x^m = 0]$$

$$\Rightarrow \quad (m^2 + 4m)\, a_0 = 0$$

$$\Rightarrow \quad m(m+4) = 0 \qquad \text{(Indicial equation)} \qquad (\because\ a_0 \neq 0)$$

$$\Rightarrow \quad \boxed{m = 0, -4}$$

NOTES

Hence *the roots are distinct and differing by an integer*. Equating to zero, the coefficients of successive powers of x, we get

$$\text{Coefficient of } x^{m+1} = 0$$

$$(m+1)\,m\,a_1 + 5(m+1)\,a_1 = 0$$

$$\Rightarrow \quad (m+5)(m+1)a_1 = 0 \quad \Rightarrow \quad \boxed{a_1 = 0} \qquad \ldots(5)$$

$$| \because \quad m \neq -5, -1$$

$$\text{Coefficient of } x^{m+2} = 0$$

$$(m+2)(m+1)\,a_2 + 5(m+2)a_2 + a_0 = 0$$

$$(m+2)(m+6)\,a_2 + a_0 = 0$$

$$\boxed{a_2 = \frac{-a_0}{(m+2)(m+6)}} \qquad \ldots(6)$$

Again, $\quad$ Coefficient of $x^{m+3} = 0$

$$(m+3)(m+2)\,a_3 + 5(m+3)\,a_3 + a_1 = 0$$

$$(m+3)(m+7)\,a_3 + a_1 = 0$$

$$\Rightarrow \quad a_3 = \frac{-a_1}{(m+3)(m+7)}$$

$$\Rightarrow \quad \boxed{a_3 = 0} \qquad \ldots(7)$$

Similarly, $\quad a_5 = a_7 = a_9 = \ldots = 0$

Now, $\quad$ Coefficient of $x^{m+4} = 0$

$$(m+4)(m+3)\,a_4 + 5(m+4)\,a_4 + a_2 = 0$$

$$\Rightarrow \quad (m+4)(m+8)\,a_4 = -a_2$$

$$a_4 = \frac{-a_2}{(m+4)(m+8)} = \frac{a_0}{(m+2)(m+4)(m+6)(m+8)} \text{ etc.} \qquad \ldots(8)$$

These give $\quad y = a_0 x^m \left[1 - \dfrac{x^2}{(m+2)(m+6)} + \dfrac{x^4}{(m+2)(m+4)(m+6)(m+8)} - \ldots\right]$ $\ldots(9)$

Putting $\quad m = 0$ in (9), we get

$$y_1 = (y)_{m=0} = a_0 \left[1 - \frac{x^2}{2.6} + \frac{x^4}{2.4.6.8} - \ldots\right] \qquad \ldots(10)$$

If we put $m = -4$ in the series given by eqn. (9), the coefficients become infinite. To avoid this difficulty, we put $a_0 = b_0\,(m+4)$, so that

$$y = b_0 x^m \left[(m+4) - \frac{(m+4)\,x^2}{(m+2)(m+6)} + \frac{x^4}{(m+2)(m+6)(m+8)} - \ldots\right] \qquad \ldots(11)$$

Now, $\dfrac{\partial y}{\partial m} = y \log x + b_0 x^m \left[1 + \dfrac{m^2+8m+20}{(m^2+8m+12)^2}x^2 - \dfrac{(3m^2+32m+76)}{(m^3+16m^2+76m+96)^2}x^4 + \ldots\right]$

Second solution is given by

$$y_2 = \left(\frac{\partial y}{\partial m}\right)_{m=-4} = (y)_{m=-4} \log x + b_0\,x^{-4}\left(1 + \frac{x^2}{4} - \frac{x^4}{4} + \ldots\right)$$

$$= b_0 x^{-4} \log x \left[0 - 0 + \frac{x^4}{(-2)(2)(4)} - \frac{x^6}{16} + \ldots\right] + b_0\, x^{-4}\left(1 + \frac{x^2}{4} - \frac{x^4}{4} + \ldots\right)$$

$$= b_0 x^{-4} \log x \left(\frac{-x^4}{16} - \frac{x^6}{16} - \ldots\right) + b_0 x^{-4}\left(1 + \frac{x^2}{4} - \frac{x^4}{4} + \ldots\right)$$

NOTES

Hence the complete solution is given by

$$y = c_1 y_1 + c_2\, y_2$$

$$= c_1\, a_0 \left(1 - \frac{x^2}{12} + \frac{x^4}{384} - \ldots\right) + c_2\, b_0 x^{-4} \log x \left(-\frac{x^4}{16} - \frac{x^6}{16} - \ldots\right)$$

$$+ c_2 b_0 x^{-4}\left(1 + \frac{x^2}{4} - \frac{x^4}{4} + \ldots\right)$$

$$\therefore \qquad y = \text{A}\left(1 - \frac{x^2}{12} + \frac{x^4}{384} - \ldots\right) + \text{B}x^{-4}\left(1 + \frac{x^2}{4} - \frac{x^4}{4} + \ldots\right)$$

$$- \text{B} \log x \left(\frac{1}{16} + \frac{x^2}{16} + \ldots\right)$$

where $\text{A} = c_1 a_0$ and $\text{B} = c_2 b_0$.

EXERCISE D

Solve in series:

1. $x(1-x)\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} - y = 0$

2. $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (x^2 - 1)\,y = 0$

(*Bessel's equation of order one*)

3. $(x + x^2 + x^3)\dfrac{d^2y}{dx^2} + 3x^2\dfrac{dy}{dx} - 2y = 0$

4. $x(1-x)\dfrac{d^2y}{dx^2} - (1 + 3x)\dfrac{dy}{dx} + y = 0$.

Answers

1. $y = (\text{A} + \text{B} \log x)\,(x + 2x^2 + 3x^3 + 4x^4 + \ldots) + \text{B}\,(1 + x + x^2 + x^3 + \ldots)$

2. $y = \text{A}x\left(1 - \dfrac{x^2}{2.4} + \dfrac{x^4}{2.4^2.6} - \ldots\right) + \text{B}x^{-1} \log x \left(-\dfrac{x^2}{2} + \dfrac{x^4}{2^2.4} - \ldots\right)$

$$+ \text{B}x^{-1}\left[1 + \frac{x^2}{2^2} - \frac{3}{2^2.2^3}x^4 + \ldots\right]$$

3. $y = \text{A}x\left[1 + x - \dfrac{1}{2}x^2 - \dfrac{1}{2}x^3 + \ldots\right] + \text{B} \log x\,(2x + 2x^2 - x^3 + \ldots) + \text{B}(1 - x - 5x^2 - x^3 + \ldots)$

4. $y = (\text{A} + \text{B} \log x)\,(1.2\,x^2 + 2.3x^3 + 3.4x^4 + \ldots) + \text{B}\,(-1 + x + 5x^2 + 11x^3 + \ldots)$.

Case IV. When Roots are Distinct, Differ by Integer and Making One or More Coefficients Indeterminate

Let the roots be m_1 and m_2. If one of the coefficients (suppose a_1) become indeterminate when $m = m_2$, the *complete solution is given by putting* $m = m_2$ *in* y which then contains two arbitrary constants.

Note. The result contained by putting $m = m_1$ in y merely gives a numerical multiple of one of the series contained in the first solution. Hence we reject the solution obtained by putting $m = m_1$.

SOLVED EXAMPLES

Example 13. *Solve in series the differential equation: $xy'' + 2y' + xy = 0$.*

Sol. Comparing the given equation with the form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\, y = 0, \text{ we get}$$

$$P(x) = \frac{2}{x} \quad \text{and} \quad Q(x) = 1$$

At $x = 0$, $P(x)$ is not analytic $\therefore$ $x = 0$ is a *singular point.*

Also, $xP(x) = 2$ and $x^2\, Q(x) = x^2$

At $x = 0$, since $x\, P(x)$ and $x^2\, Q(x)$ are analytic $\therefore$ $x = 0$ is a *regular singular point.*

Let us assume

$$y = a_0\, x^m + a_1 x^{m+1} + a_2\, x^{m+2} + a_3\, x^{m+3} + \ldots \qquad \ldots(1)$$

Then, $$\frac{dy}{dx} = ma_0\, x^{m+1} + (m+1)\, a_1\, x^m + (m+2)\, a_2\, x^{m+1} + (m+3)\, a_3\, x^{m+2} + \ldots$$

and $$\frac{d^2y}{dx^2} = m\,(m-1)\, a_0\, x^{m-2} + (m+1)\, m\, a_1\, x^{m-1} + (m+2)\,(m+1)\, a_2\, x^m + (m+3)\,(m+2)\, a_3 x^{m+1} + \ldots$$

Substituting these values in the given equation, we get

$$x\,[m\,(m-1)\, a_0\, x^{m-2} + (m+1)\, m\, a_1\, x^{m-1} + (m+2)\,(m+1)\, a_2 x^m + (m+3)\,(m+2)\, a_3\, x^{m+1} + \ldots]$$
$$+ 2\,[m\, a_0\, x^{m-1} + (m+1)\, a_1\, x^m + (m+2)\, a_2\, x^{m+1} + \ldots]$$
$$+ x\,[a_0\, x^m + a_1\, x^{m+1} + a_2\, x^{m+2} + a_3\, x^{m+3} + \ldots] = 0$$

Now, Coefficient of $x^{m-1} = 0$

$\Rightarrow$ $m\,(m-1)\, a_0 + 2m\, a_0 = 0$

$(m^2 + m)\, a_0 = 0$

$\Rightarrow$ $m^2 + m = 0$ (Indicial equation) $\mid \because\ a_0 \neq 0$

$\Rightarrow$ $\boxed{m = 0, -1}$

Hence *roots are distinct and differ by an integer.*

Coefficient of $x^m = 0$

$\Rightarrow$ $(m+1)\, m\, a_1 + 2(m+1)\, a_1 = 0$

$\Rightarrow$ $(m+1)\,(m+2)a_1 = 0$

$\Rightarrow$ $(m+1)a_1 = 0$ $\mid \because\ m + 2 \neq 0$

Since $m + 1$ may be zero, hence a_1 *is arbitrary* (or takes the form $\frac{0}{0}$). In other words, a_1 becomes indeterminate.

Hence the solution will contain a_0 and a_1 as arbitrary constants. The complete solution will be given by putting $m = -1$ in y.

Now, Coefficient of $x^{m+1} = 0$

$\Rightarrow$ $(m+2)\,(m+1)\, a_2 + 2(m+2)\, a_2 + a_0 = 0$

$\Rightarrow$ $(m+2)\,(m+3)\, a_2 + a_0 = 0$

$$\boxed{a_2 = \frac{-a_0}{(m+2)(m+3)}}$$

Coefficient of $x^{m+2} = 0$

$$\Rightarrow \quad (m+3)(m+2)a_3 + 2(m+3)a_3 + a_1 = 0$$

$$(m+3)(m+4)a_3 + a_4 = 0$$

$$\boxed{a_3 = \frac{-a_1}{(m+3)(m+4)}}$$

NOTES

Coefficient of $x^{m+3} = 0$

$$\Rightarrow \quad (m+4)(m+3)a_4 + 2(m+4)a_4 + a_2 = 0$$

$$\Rightarrow \quad (m+4)(m+5)a_4 = -a_2$$

$$\Rightarrow \quad a_4 = \frac{-a_2}{(m+4)(m+5)}$$

$$\Rightarrow \quad \boxed{a_4 = \frac{a_0}{(m+2)(m+3)(m+4)(m+5)}}$$

Coefficient of $x^{m+4} = 0$

$$(m+5)(m+4)a_5 + 2(m+5)a_5 + a_3 = 0$$

$$(m+5)(m+6)a_5 = -a_3$$

$$\boxed{a_5 = \frac{a_1}{(m+3)(m+4)(m+5)(m+6)}}$$

and so on.

Substituting these values in eqn. (1), we get

$$y = x^m \left[a_0 + a_1 x - \frac{a_0}{(m+2)(m+3)}x^2 - \frac{a_1}{(m+3)(m+4)}x^3 + \frac{a_0}{(m+2)(m+3)(m+4)(m+5)}x^4 \right.$$
$$\left. + \frac{a_1}{(m+3)(m+4)(m+5)(m+6)}x^5 + \ldots\right]$$

$$y = x^m \left[a_0 \left\{1 - \frac{x^2}{(m+2)(m+3)} + \frac{x^4}{(m+2)(m+3)(m+4)(m+5)} - \ldots\right\}\right.$$
$$\left. + a_1 \left\{x - \frac{x^3}{(m+3)(m+4)} + \frac{x^5}{(m+3)(m+4)(m+5)(m+6)} - \ldots\right\}\right]$$

Now, $$(y)_{m=-1} = x^{-1}\left[a_0\left(1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \ldots\right) + a_1\left(x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \ldots\right)\right]$$

$$= x^{-1}[a_0 \cos x + a_1 \sin x]$$

Hence complete solution is given by

$$y = (y)_{m=-1}$$

$$\Rightarrow \quad y = \frac{1}{x}(a_0 \cos x + a_1 \sin x).$$

Note. All those problems, in which $x = 0$ was an ordinary point of $y'' + P(x)y' + Q(x)y = 0$, can also be solved by Frobenius method as given in *Art. 2.4.4* and explained in above illustrative example.

EXERCISE F

Solve in series:

1. $x^2 \dfrac{d^2y}{dx^2} + 4x \dfrac{dy}{dx} + (x^2 + 2)\, y = 0$

2. $(1 - x^2) \dfrac{d^2y}{dx^2} - x \dfrac{dy}{dx} + 4y = 0$

3. $(1 - x^2) \dfrac{d^2y}{dx^2} - 2x \dfrac{dy}{dx} + n(n + 1)\, y = 0.$

Answers

1. $y = x^{-2} (a_0 \cos x + a_1 \sin x)$

2. $y = a_0 (1 - 2x^2) + a_1 \left(x - \dfrac{x^3}{2} - \dfrac{x^5}{8} + \dfrac{x^7}{16} - \ldots \right)$

3. $$y = a_0 \left[1 - \frac{n(n+1)}{2!} x^2 + \frac{(n-2)\, n(n+1)(n+3)}{4!} x^4 - \ldots \right] + a_1 \left[x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} x^5 + \ldots \right]$$

6. DIFFERENTIAL EQUATIONS

STRUCTURE

INTRODUCTION

The differential equation $$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n + 1)y = 0 \qquad ...(1)$$
where n is real number, is called Legendre's differential equation. This equation is of considerable importance in applied mathematics, particularly in boundary value problems involving spherical configurations.

Though n is a real number, in most physical applications, only integral values of n are required. Also, equation (1) can be solved in series of ascending or descending powers of x. The solution in descending powers of x is more important than the one in ascending powers.

Let $$y = \sum_{k=0}^{\infty} a_k x^{m-k}$$

then $$\frac{dy}{dx} = \sum_{k=0}^{\infty} (m - k)\, a_k x^{m-k-1} \quad \text{and} \quad \frac{d^2y}{dx^2} = \sum_{k=0}^{\infty} (m - k)(m - k - 1)\, a_k x^{m-k-2}$$

NOTES

Substituting for y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1), we get

$$(1 - x^2) \sum_{k=0}^{\infty} (m-k)(m-k-1)\, a_k\, x^{m-k-2} - 2x \sum_{k=0}^{\infty} (m-k) a_k\, x^{m-k-1} + n(n+1) \sum_{k=0}^{\infty} a_k\, x^{m-k} = 0$$

or
$$\sum_{k=0}^{\infty} (m-k)(m-k-1)\, a_k\, x^{m-k-2} - \sum_{k=0}^{\infty} [(m-k)(m-k-1) + 2(m-k) - n(n+1)] a_k\, x^{m-k} = 0$$

or
$$\sum_{k=0}^{\infty} (m-k)(m-k-1)\, a_k\, x^{m-k-2} - \sum_{k=0}^{\infty} (m-k)^2 - n^2 + (m-k) - n]\, a_k\, x^{m-k} = 0$$

or
$$\sum_{k=0}^{\infty} (m-k)(m-k-1)\, a_k\, x^{m-k-2} - \sum_{k=0}^{\infty} [(m-k-n)\,(m-k+n+1)\, a_k\, x^{m-k} = 0.$$

Equating to zero the coefficient of highest power of x, *i.e.*, x^m, we get the indicial equation

$$(m-n)\,(m+n+1)\, a_0 = 0$$

whence $m = n$ or $m = -(n+1)$ since $a_0 \neq 0$

Equating to zero the coefficient of the next lower power of x, *i.e.*, x^{m-1}, we get

$$(m+n)\,(m-n-1)\, a_1 = 0 \text{ or } a_1 = 0,$$

since $(m+n)$ and $(m-n-1)$ are not zero for $m = n$ or $-(n+1)$.

Equating to zero the coefficient of x^{m-k}, we get the recurrence relation

$$[m-(k-2)]\,[m-(k-2)-1]\, a_{k-2} - (m-k-n)\,(m-k+n+1)\, a_k = 0$$

or
$$\boxed{a_k = -\frac{(m-k+2)(m-k+1)}{(n-m+k)(n+m-k+1)}\, a_{k-2}} \qquad ...(2)$$

Since $a_1 = 0$, therefore, from (2), we get $a_3 = a_5 = a_7 = ... = 0$.

Case I. *When* $m = n$, the recurrence relation (2) reduces to

$$a_k = -\frac{(n-k+2)(n-k+1)}{k(2n-k+1)}\, a_{k-2}$$

Putting $k = 2, 4, 6, ...$, we get $a_2 = -\frac{n(n-1)}{2(2n-1)} a_0$,

$$a_4 = -\frac{(n-2)(n-3)}{4(2n-3)}\, a_2 = \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}\, a_0, \text{ etc.}$$

Therefore, one solution of Legendre's equation is given by

$$y_1 = a_0 \left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)} x^{n-4} - ... \right] \qquad ...(3)$$

Case II. *When* $m = -(n+1)$, the recurrence relation (2) reduces to

$$a_k = \frac{(n+k-1)(n+k)}{k(2n+k+1)}\, a_{k-2}$$

Putting $k = 2, 4, 6, \ldots\ldots$, we get

$$a_2 = \frac{(n+1)(n+2)}{2(2n+3)} a_0,$$

$$a_4 = \frac{(n+3)(n+4)}{4(2n+5)} a_2 = \frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)} a_0, \text{ etc.}$$

Therefore, the second solution of Legendre's equation is given by

$$y_2 = a_0 \left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)} x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)} x^{-n-5} + \ldots \right] \quad \ldots(4)$$

LEGENDRE'S FUNCTION OF FIRST KIND $P_n(x)$

When n is a positive integer and $a_0 = \frac{1.3.5\ldots\ldots(2n-1)}{n!}$,

the first solution given by (3) is denoted by $P_n(x)$ and is called Legendre's function of first kind.

Thus,

$$P_n(x) = \frac{1.3.5\ldots\ldots(2n-1)}{n!} \left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1).(n-2)(n-3)}{2.4.(2n-1)(2n-3)} x^{n-4} - \ldots \right]$$

$P_n(x)$ is a terminating series. RHS is known as Zonal Harmonic. $P_n(x)$ gives Legendre's polynomials for different values of n such that $P_n(1) = 1$.

Now, two cases arise:

Case I. *When n is even:*

No. of terms in the series within bracket $= \frac{n}{2} + 1$

$$\text{Last term} = (-1)^{n/2} \cdot \frac{\{n(n-1)(n-2)(n-3)\ldots\ldots 2.1\}}{(2.4.6\ldots n)\{(2n-1)(2n-3)\ldots(n+1)\}}.$$

Case II. *When n is odd:*

No. of terms in the series within bracket $= \frac{n+1}{2}$

$$\text{Last term} = (-1)^{\frac{n-1}{2}} \cdot \frac{n(n-1)(n-2)(n-3)\ldots 3.2}{\{2.4.6\ldots(n-1)\}\{(2n-1)(2n-3)\ldots(n+2)\}}.$$

LEGENDRE'S FUNCTION OF SECOND KIND $Q_n(x)$

When n is a positive integer and $a_0 = \frac{n!}{1.3.5\ldots(2n+1)}$,

the second solution is denoted by $Q_n(x)$ and is called Legendre's function of second kind.

NOTES

Thus, $$Q_n(x) = \frac{n!}{1.3.5\ldots(2n+1)}\left[x^{-n-1} + \frac{(n+1)(n+2)}{2.(2n+3)}x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot4\cdot(2n+3)(2n+5)}x^{-n-5} + \cdots\right]$$

It is a non-terminating series so there is no last term.

SOLUTION OF LEGENDRE'S EQUATION

Since $y = P_n(x)$ and $y = Q_n(x)$ both are the solutions of the given equation hence the most general solution is given by

$$y = AP_n(x) + BQ_n(x)$$

where A and B are arbitrary constants.

GENERATING FUNCTION FOR $P_n(x)$

We shall show that $P_n(x)$ is the coefficient of h^n in the expansion of $(1 - 2xh + h^2)^{-1/2}$ in ascending powers of h.

i.e., $$(1 - 2xh + h^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) \,.\, h^n$$

Using Binomial theorem,

$$(1-t)^{-1/2} = 1 + \frac{1}{2}t + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}t^2 + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!}t^3 + \cdots$$

$$= 1 + \frac{1}{2}t + \frac{1.3}{2.4}t^2 + \frac{1.3.5}{2.4.6}t^3 + \cdots + \frac{1.3.5\ldots\ldots(2n-1)}{2.4.6\ldots\ldots 2n}t^n + \cdots$$

$$\therefore \quad (1 - 2xh + h^2)^{-1/2} = [1 - h(2x-h)]^{-1/2}$$

$$= 1 + \frac{1}{2}h(2x-h) + \frac{1.3}{2.4}h^2(2x-h)^2 + \cdots$$
$$+ \frac{1.3.5\ldots\ldots(2n-5)}{2.4.6\ldots\ldots(2n-4)}h^{n-2}(2x-h)^{n-2}$$
$$+ \frac{1.3.5\ldots\ldots(2n-3)}{2.4.6\ldots\ldots(2n-2)}h^{n-1}(2x-h)^{n-1}$$
$$+ \frac{1.3.5\ldots\ldots(2n-1)}{2.4.6\ldots\ldots(2n)}h^n(2x-h)^n + \cdots$$

Now, the coefficient of h^n in $\frac{1.3.5\ldots\ldots(2n-1)}{2.4.6\ldots\ldots(2n)}h^n(2x-h)^n$ is

$$= \frac{1.3.5\ldots\ldots(2n-1)}{2.4.6\ldots\ldots(2n)}(2x)^n = \frac{1.3.5\ldots\ldots(2n-1)}{2^n(n)!}(2x)^n = \frac{1.3.5\ldots\ldots(2n-1)}{n!}x^n$$

The coefficient of h^n in $\frac{1.3.5......(2n-3)}{2.4.6......(2n-2)}\ h^{n-1}\ (2x-h)^{n-1}$ is

$$= \frac{1.3.5......(2n-3)}{2.4.6......(2n-2)}\ [-\ {}^{n-1}C_1\ (2x)^{n-2}] = -\frac{1.3.5......(2n-3)}{2.4.6......(2n-2)}\ (n-1)\ 2^{n-2}\ .\ x^{n-2}$$

$$= -\frac{1.3.5......(2n-3)}{2^{n-1}\ (n-1)!}\ (n-1)\ 2^{n-2}\ .\ x^{n-2}$$

$$= -\frac{1.3.5......(2n-3)(2n-1)}{2\,(n)!} \cdot \frac{n(n-1)}{2n-1}\ x^{n-2}$$

$$= -\frac{1.3.5......(2n-1)}{n!} \cdot \frac{n(n-1)}{2(2n-1)}\ x^{n-2}$$

Similarly, the coefficient of h^n in $\frac{1.3.5......(2n-5)}{2.4.6......(2n-4)}\ h^{n-2}\ (2x-h)^{n-2}$ is

$$= \frac{1.3.5......(2n-1)}{n!} \cdot \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}\ x^{n-4} \text{ and so on.}$$

$\therefore$ The coefficient of h^n in $(1-2xh+h^2)^{-1/2}$ is given by

$$\frac{1.3.5......(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}x^{n-4} - ...\right] = P_n(x)$$

Thus, in the expansion of $(1-2xh+h^2)^{-1/2}$, $P_1(x)$, $P_2(x)$, $P_3(x)$, ..., $P_n(x)$, ... are the coefficients of h, h^2, h^3, ..., h^n, ... respectively.

$$\therefore \quad (1-2xh+h^2)^{-1/2} = 1 + P_1(x)\ .\ h + P_2(x)\ .\ h^2 + ... + P_n(x)\ .\ h^n + ... = \sum_{n=0}^{\infty} P_n(x)\ .\ h^n$$

The function $(1-2xh+h^2)^{-1/2}$ is called the **generating function for $P_n(x)$.**

SOLVED EXAMPLES

Example 1. *Show that*

(i) $P_n(1) = 1$ *(ii)* $P_n(-x) = (-1)^n P_n(x)$ *(iii)* $P'_n(-x) = (-1)^{n+1} P'_n(x)$.

Sol. We know that $$\sum_{n=0}^{\infty} h^n\ P_n(x) = (1-2xh+h^2)^{-1/2} \quad ...(1)$$

(i) Putting $x = 1$ in eqn. (1), we get

$$\sum_{n=0}^{\infty} h^n\ P_n(1) = (1-2h+h^2)^{-1/2} = (1-h)^{-1}$$

$$= 1 + h + h^2 + ... + h^n + ... = \sum_{n=0}^{\infty} h^n$$

Equating the coefficients of h^n, we have $P_n(1) = 1$.

(ii) Replacing x by $(-x)$ in eqn. (1), we get

$$\sum_{n=0}^{\infty} h^n P_n(-x) = (1+2xh+h^2)^{-1/2} \quad ...(2)$$

Again, replacing h by $(-h)$ in eqn. (1), we have

$$\sum_{n=0}^{\infty} (-h)^n\ P_n(x) = (1+2xh+h^2)^{-1/2}$$

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or $$\sum_{n=0}^{\infty} (-1)^n h^n P_n(x) = (1 + 2xh + h^2)^{-1/2} \quad ...(3)$$

From (2) and (3), $\sum_{n=0}^{\infty} h^n P_n(-x) = \sum_{n=0}^{\infty} (-1)^n h^n P_n(x)$

Equating the coefficients of h^n, we have

$$P_n(-x) = (-1)^n P_n(x).$$

(*iii*) We have, $P_n(-x) = (-1)^n P_n(x)$ | Proved in (*ii*)

Differentiating w.r.t. x, we get

$$-P'_n(-x) = (-1)^n P'_n(x)$$

$$\therefore \quad P'_n(-x) = (-1)^{n+1} P'_n(x).$$

Example 2. *Show that:*

(*i*) $P_{2n}(0) = (-1)^n \dfrac{2n!}{2^{2n}(n!)^2}$ (*ii*) $P_{2n+1}(0) = 0.$

Sol. We know that $\sum_{n=0}^{\infty} h^n P_n(x) = (1 - 2xh + h^2)^{-1/2}$

Putting $x = 0$, we get $\sum_{n=0}^{\infty} h^n P_n(0) = (1 + h^2)^{-1/2}$

$$= 1 - \frac{1}{2}h^2 + \frac{1.3}{2.4}h^4 - \ldots\ldots + (-1)^r \cdot \frac{1.3.5\ldots\ldots(2r-1)}{2.4.6\ldots\ldots(2r)} h^{2r} + \cdots$$

(*i*) Equating the coefficients of h^{2n} on both sides, we get

$$P_{2n}(0) = (-1)^n \frac{1.3.5\ldots\ldots(2n-1)}{2.4.6\ldots\ldots(2n)} = (-1)^n \frac{1.2.3.4\ldots\ldots(2n-1)(2n)}{[2.4.6\ldots\ldots(2n)]^2}$$

$$= (-1)^n \frac{(2n)!}{[2^n\, 1.2.3\ldots\ldots n]^2} = (-1)^n \frac{(2n)!}{2^{2n}(n!)^2}.$$

(*ii*) Equating the coefficients of h^{2n+1} on both sides, we get $P_{2n+1}(0) = 0$, since the right-hand side contains only even power of h.

Example 3. *Prove that:*

(*i*) $\sum_{n=0}^{\infty} P_n(x) = \dfrac{1}{\sqrt{2-2x}}$ (*ii*) $P_n(-1) = (-1)^n.$

Sol. We know that,

$$(1 - 2xh + h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x) \quad ...(1)$$

(*i*) Put $h = 1$ in (1), we get

$$(1 - 2x + 1)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)$$

$$\Rightarrow \quad \frac{1}{\sqrt{2-2x}} = \sum_{n=0}^{\infty} P_n(x)$$

(*ii*) We have already proved in example 1 (*ii*) that

$$P_n(-x) = (-1)^n P_n(x)$$

Put $x = 1$ in above relation, we get

$$P_n(-1) = (-1)^n P_n(1) = (-1)^n \qquad | \because \quad P_n(1) = 1$$

Example 4. *Prove that:*

$$\frac{1-z^2}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty} (2n+1) P_n z^n .$$

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Sol. We know that

$$(1 - 2zx + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x) \qquad ...(1)$$

Differentiating (1) w.r.t. z, we get

$$-\frac{1}{2} (1 - 2zx + z^2)^{-3/2} . (2z - 2x) = \sum_{n=0}^{\infty} nz^{n-1} P_n(x)$$

$$\Rightarrow \quad (x - z) (1 - 2zx + z^2)^{-3/2} = \sum_{n=0}^{\infty} nz^{n-1} P_n(x) \qquad ...(2)$$

Multiplying both sides of eqn. (2) by $2z$, we get

$$2z (x - z) (1 - 2zx + z^2)^{-3/2} = \sum_{n=0}^{\infty} 2nz^n P_n(x) \qquad ...(3)$$

Adding (1) and (3), we get

$$(1 - 2zx + z^2)^{-3/2} (2zx - 2z^2 + 1 - 2zx + z^2) = \sum_{n=0}^{\infty} (2n+1) z^n P_n(x)$$

$$\Rightarrow \quad \frac{1-z^2}{(1-2zx+z^2)^{3/2}} = \sum_{n=0}^{\infty} (2n+1) z^n P_n(x).$$

Example 5. *Prove that:* $\dfrac{1+z}{z\sqrt{1-2xz+z^2}} - \dfrac{1}{z} = \sum_{n=0}^{\infty} (P_n + P_{n+1}) z^n .$

Sol. RHS $= \sum_{n=0}^{\infty} P_n z^n + \sum_{n=0}^{\infty} P_{n+1} z^n$

$$= \sum_{n=0}^{\infty} P_n z^n + \frac{1}{z} \sum_{n=0}^{\infty} P_{n+1} z^{n+1}$$

$$= \sum_{n=0}^{\infty} P_n z^n + \frac{1}{z} (P_1 z + P_2 z^2 + P_3 z^3 + ... + P_n z^n + ...)$$

$$= \sum_{n=0}^{\infty} P_n z^n + \frac{1}{z} \{- P_0 + P_0 z^0 + P_1 z + P_2 z^2 + ... + P_n z^n + ...\} \quad | \because \quad P_0 = 1$$

$$= \sum_{n=0}^{\infty} P_n z^n - \frac{1}{z} + \frac{1}{z} \sum_{n=0}^{\infty} P_n z^n = \left(1 + \frac{1}{z}\right) \sum_{n=0}^{\infty} P_n z^n - \frac{1}{z} \qquad ...(1)$$

$$\text{LHS} = \frac{1+z}{z\sqrt{1-2xz+z^2}} - \frac{1}{z} = \left(1 + \frac{1}{z}\right) (1 - 2xz + z^2)^{-1/2} - \frac{1}{z}$$

$$= \left(1 + \frac{1}{z}\right) \sum_{n=0}^{\infty} z^n P_n - \frac{1}{z} \qquad ...(2)$$

Hence the proof. | Since LHS = RHS

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Example 6. *Prove that:*

(*i*) $P_n'(1) = \dfrac{n(n+1)}{2}$

(*ii*) $P_n'(-1) = (-1)^{n-1} \cdot \dfrac{n(n+1)}{2}$ *Or* $P_n'(-1) = (-1)^{n+1} \cdot \dfrac{n(n+1)}{2}$.

Sol. Legendre's differential equation is

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)\,y = 0$$

$y = P_n(x)$ is a solution of it. So,

$$(1-x^2)\,P_n''(x) - 2x\,P_n'(x) + n(n+1)\,P_n(x) = 0 \qquad ...(1)$$

(*i*) Put $x = 1$ in (1), $\quad -2P_n'(1) + n(n+1)\,P_n(1) = 0$

$$\Rightarrow \qquad P_n'(1) = \frac{n(n+1)}{2} \qquad \mid \because \; P_n(1) = 1$$

(*ii*) Put $x = -1$ in (1),

$$2P_n'(-1) + n(n+1)\,P_n(-1) = 0$$

$$\Rightarrow \qquad 2P_n'(-1) + n(n+1)(-1)^n = 0 \qquad \mid \text{See Ex. 3 } (ii)$$

$$\Rightarrow \qquad 2P_n'(-1) = -n(n+1)(-1)^n$$

Case I. $\quad 2P_n'(-1) = (-1)^1 \cdot n(n+1)(-1)^n = (-1)^{n+1} \cdot n(n+1)$

$$\Rightarrow \qquad P_n'(-1) = (-1)^{n+1} \cdot \frac{n(n+1)}{2}.$$

Case II. $\quad 2P_n'(-1) = (-1)^{-1} \cdot n(n+1) \cdot (-1)^n = (-1)^{n-1} \cdot n(n+1)$

$$\Rightarrow \qquad P_n'(-1) = (-1)^{n-1} \cdot \frac{n(n+1)}{2}$$

RODRIGUE'S FORMULA

The relation $\quad P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}(x^2-1)^n$ is known as Rodrigue's formula.

To prove it, let $\quad v = (x^2-1)^n$ then $\quad v_1 = \dfrac{dv}{dx} = n(x^2-1)^{n-1} \cdot 2x$

Multiplying both sides by $(x^2 - 1)$, we get

$$(x^2-1)v_1 = 2nx\,(x^2-1)^n = 2nxv$$

or

$$(1-x^2)v_1 + 2nxv = 0$$

Differentiating $(n + 1)$ times by Leibnitz's theorem, we have

$$\left[(1-x^2)v_{n+2} + (n+1)(-2x)v_{n+1} + \frac{(n+1)n}{2!}(-2)v_n\right] + 2n\,[xv_{n+1} + (n+1)\,v_n] = 0$$

or

$$(1-x^2)v_{n+2} - 2xv_{n+1} + n(n+1)v_n = 0$$

or

$$(1-x^2)\frac{d^2(v_n)}{dx^2} - 2x\frac{d(v_n)}{dx} + n(n+1)v_n = 0$$

which is Legendre's equation and v_n is its solution. But the solutions of Legendre's equations are $P_n(x)$ and $Q_n(x)$.

Since $v_n = \dfrac{d^n}{dx^n}(x^2-1)^n$ contains only positive powers of x, it must be a constant multiple of $P_n(x)$.

i.e., $\quad v_n = c\mathrm{P}_n(x)$

or $\quad c\mathrm{P}_n(x) = \dfrac{d^n}{dx^n}(x^2-1)^n$

$$= \frac{d^n}{dx^n}[(x-1)^n (x+1)^n] \qquad \ldots(1)$$

$$= (x-1)^n \frac{d^n}{dx^n}(x+1)^n + {}^n\mathrm{C}_1 \,.\, n(x-1)^{n-1} \frac{d^{n-1}}{dx^{n-1}}(x+1)^n + \cdots\cdots + (x+1)^n \frac{d^n}{dx^n}(x-1)^n$$

$$= (x-1)^n . n\,! + {}^n\mathrm{C}_1 . n(x-1)^{n-1} \,.\, \frac{d^{n-1}}{dx^{n-1}}(x+1)^n + \cdots\cdots + (x+1)^n\, n\,!$$

$$= n\,!\,(x+1)^n + \text{terms containing powers of } (x-1)$$

Putting $x = 1$ on both sides, we get

$c\mathrm{P}_n(1) = n\,!\,.\,2^n$ or $c = 2^n\, n\,!$, since $\mathrm{P}_n(1) = 1$

Substituting in (1), we get

$$\boxed{\mathrm{P}_n(x) = \frac{1}{2^n\, n\,!}\frac{d^n}{dx^n}(x^2-1)^n}$$

Putting $n = 0, 1, 2, 3, \ldots\ldots$ in Rodrigue's formula, we get Legendre's polynomials. Thus

$$\mathrm{P}_0(x) = 1$$

$$\mathrm{P}_1(x) = \frac{1}{2}\frac{d}{dx}(x^2-1) = x$$

$$\mathrm{P}_2(x) = \frac{1}{2^2(2)!}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{8}\frac{d^2}{dx^2}(x^4-2x^2+1) = \tfrac{1}{2}(3x^2-1)$$

$$\mathrm{P}_3(x) = \frac{1}{2^3(3)!}\frac{d^3}{dx^3}(x^2-1)^3 = \frac{1}{48}\frac{d^3}{dx^3}(x^6-3x^4+3x^2-1)$$

$$= \tfrac{1}{2}(5x^3-3x)$$

Similarly, $\quad \mathrm{P}_4(x) = \dfrac{1}{8}(35x^4-30x^2+3)$

$$\mathrm{P}_5(x) = \frac{1}{8}(63x^5-70x^3+15x)$$

$$\mathrm{P}_6(x) = \frac{1}{6}(231x^6-351x^4+105x^2-5) \text{ etc.}$$

SOLVED EXAMPLES

Example 7. *Show that* $x^4 = \dfrac{1}{35}[8P_4(x) + 20P_2(x) + 7P_0(x)]$.

Sol. We know that $\quad \mathrm{P}_4(x) = \tfrac{1}{8}(35x^4-30x^2+3)$

$$\mathrm{P}_2(x) = \tfrac{1}{2}(3x^2-1),\ \mathrm{P}_0(x) = 1$$

$$\therefore \quad \frac{1}{35}[8\mathrm{P}_4(x) + 20\mathrm{P}_2(x) + 7\mathrm{P}_0(x)] = \frac{1}{35}[35x^4-30x^2+3) + 10(3x^2-1) + 7] = x^4.$$

Example 8. *Express $f(x) = x^3 - 5x^2 + x + 2$ in terms of Legendre's polynomials.*

Sol. We know that $\quad P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$$\therefore \quad x^3 = \frac{2}{5} P_3(x) + \frac{3}{5} x$$

$$\therefore \quad f(x) = \left[\frac{2}{3} P_3(x) + \frac{3}{5}(x)\right] - 5x^2 + x + 2 = \frac{2}{5} P_3(x) - 5x^2 + \frac{8}{5} x + 2$$

$$= \frac{2}{5} P_3(x) - 5\left[\frac{2}{3} P_2(x) + \frac{1}{3}\right] + \frac{8}{5} x + 2$$

$$\left[\because \quad P_2(x) = \tfrac{1}{2}(3x^2 - 1) \quad \therefore \quad x^2 = \tfrac{2}{3} P_2(x) + \tfrac{1}{3}\right]$$

$$= \frac{2}{5} P_3(x) - \frac{10}{3} P_2(x) + \frac{8}{5} x + \frac{1}{3}$$

$$= \frac{2}{5} P_3(x) - \frac{10}{3} P_2(x) + \frac{8}{5} P_1(x) + \frac{1}{3} P_0(x)$$

$$[\because \quad x = P_1(x) \text{ and } 1 = P_0(x)]$$

Example 9. *Prove that:* $\int_{-1}^{1} P_n(x)\, dx = \left.\begin{matrix} 0, & n \neq 0 \\ 2, & n = 0 \end{matrix}\right\}.$

Sol. We know by Rodrigue's formula, $P_n(x) = \frac{1}{2^n\, n!} \frac{d^n}{dx^n} \{(x^2 - 1)^n\}$

Integrating, we get

$$\int_{-1}^{1} P_n(x)\, dx = \frac{1}{2^n\, n!} \int_{-1}^{1} \frac{d^n}{dx^n} (x^2 - 1)^n\, dx$$

$$= \frac{1}{2^n\, n!} \left[\frac{d^{n-1}}{dx^{n-1}} (x^2 - 1)^n\right]_{-1}^{1} = 0$$

When $n = 0$, $\int_{-1}^{1} P_0(x)\, dx = \int_{-1}^{1} 1\, dx = 2. \qquad | \because \quad P_0(x) = 1$

Example 10. *Express $4x^3 + 6x^2 + 7x + 2$ in terms of Legendre's polynomials.*

Sol. Let $\quad 4x^3 + 6x^2 + 7x + 2 \equiv \alpha P_3(x) + \beta P_2(x) + \gamma P_1(x) + \xi P_0(x) \qquad \ldots(1)$

$$\equiv \alpha\left(\frac{5x^3 - 3x}{2}\right) + \beta\left(\frac{3x^2 - 1}{2}\right) + \gamma(x) + \xi(1)$$

$$\equiv \frac{5\alpha}{2} x^3 + \frac{3\beta}{2} x^2 + \left(\gamma - \frac{3\alpha}{2}\right) x + \left(\xi - \frac{\beta}{2}\right)$$

Equating the coefficients of like powers of x, we get

$$\frac{5\alpha}{2} = 4 \qquad \Rightarrow \quad \alpha = \frac{8}{5}$$

$$6 = \frac{3\beta}{2} \qquad \Rightarrow \quad \beta = 4$$

$$7 = \gamma - \frac{3\alpha}{2} \qquad \Rightarrow \quad 7 = \gamma - \frac{12}{5} \quad \Rightarrow \quad \gamma = \frac{47}{5}$$

$$2 = \xi - \frac{\beta}{2} \qquad \Rightarrow \quad 2 = \xi - 2 \quad \Rightarrow \quad \xi = 4$$

Hence from (1),

$$4x^3 + 6x^2 + 7x + 2 = \frac{8}{5} P_3(x) + 4P_2(x) + \frac{47}{5} P_1(x) + 4P_0(x).$$

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Example 11. *Prove that:*

$$P_n\left(-\frac{1}{2}\right) = P_0\left(-\frac{1}{2}\right)P_{2n}\left(\frac{1}{2}\right) + P_1\left(-\frac{1}{2}\right)P_{2n-1}\left(\frac{1}{2}\right) + \cdots + P_{2n}\left(-\frac{1}{2}\right)P_0\left(\frac{1}{2}\right).$$

Sol. We know that,

$$\sum_{n=0}^{\infty} h^n \, P_n(x) = (1 - 2hx + h^2)^{-1/2} \quad \ldots(1)$$

Put $x = \frac{1}{2}$ in (1),

$$\sum_{n=0}^{\infty} h^n \, P_n\left(\frac{1}{2}\right) = (1 - h + h^2)^{-1/2} \quad \ldots(2)$$

Put $x = -\frac{1}{2}$ in (1),

$$\sum_{n=0}^{\infty} h^n \, P_n\left(-\frac{1}{2}\right) = (1 + h + h^2)^{-1/2} \quad \ldots(3)$$

Replacing h by h^2 in (3), we get

$$\sum_{n=0}^{\infty} h^{2n} \, P_n\left(-\frac{1}{2}\right) = (1 + h^2 + h^4)^{-1/2} = [(1 + h^2)^2 - h^2]^{-1/2}$$

$$= (1 + h^2 + h)^{-1/2} \, (1 + h^2 - h)^{-1/2}$$

$$= \sum_{n=0}^{\infty} h^n \, P_n\left(-\frac{1}{2}\right) . \sum_{n=0}^{\infty} h^n \, P_n\left(\frac{1}{2}\right) \quad \text{| From (2) and (3)}$$

$$\Rightarrow \quad \sum_{n=0}^{\infty} h^{2n} \, P_n\left(-\frac{1}{2}\right) = \left[P_0\left(-\frac{1}{2}\right) + h\,P_1\left(-\frac{1}{2}\right) + \cdots + h^{2n-1}\,P_{2n-1}\left(-\frac{1}{2}\right) + h^{2n}\,P_{2n}\left(-\frac{1}{2}\right) + \cdots\right]$$

$$\left[P_0\left(\frac{1}{2}\right) + hP_1\left(\frac{1}{2}\right) + \cdots\cdots + h^{2n-1}\,P_{2n-1}\left(\frac{1}{2}\right) + h^{2n}\,P_{2n}\left(\frac{1}{2}\right) + \cdots\right]$$

Equating the coefficients from both sides of the above equation, we get the required result.

$$P_n\left(-\frac{1}{2}\right) = P_0\left(-\frac{1}{2}\right) . P_{2n}\left(\frac{1}{2}\right) + P_1\left(-\frac{1}{2}\right)P_{2n-1}\left(\frac{1}{2}\right) + \cdots\cdots + P_{2n}\left(-\frac{1}{2}\right)P_0\left(\frac{1}{2}\right).$$

Example 12. *Let $P_n(x)$ be the Legendre polynomial of degree n. Show that for any function f(x), for which the nth derivative is continuous*

$$\int_{-1}^{1} f(x)\,P_n(x)\,dx = \frac{(-1)^n}{2^n\,n!}\int_{-1}^{1} (x^2 - 1)^n\,f^{(n)}(x)\,dx.$$

Sol. $\int_{-1}^{1} f(x)\,P_n(x)\,dx = \int_{-1}^{1} f(x) \, . \, \frac{1}{2^n\,n!}\,\frac{d^n}{dx^n}\,(x^2 - 1)^n\,dx$ | Using Rodrigue's formula

$$= \frac{1}{2^n\,n!}\int_{-1}^{1} f(x) \, . \, \frac{d^n}{dx^n}\,(x^2 - 1)^n\,dx$$

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$$= \frac{1}{2^n n!}\left[\left\{f(x)\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\right\}_{-1}^{1} - \int_{-1}^{1} f'(x)\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\,dx\right]$$

| Integrating by parts

$$= \frac{1}{2^n n!}\left[0 - \int_{-1}^{1} f'(x)\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\,dx\right]$$

$$= \frac{(-1)^1}{2^n n!}\int_{-1}^{1} f'(x)\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\,dx$$

$$= \frac{(-1)^2}{2^n n!}\int_{-1}^{1} f''(x)\frac{d^{n-2}}{dx^{n-2}}(x^2-1)^n\,dx$$ | Integrating by parts again

$$= \frac{(-1)^n}{2^n n!}\int_{-1}^{1} f^{(n)}(x)\,(x^2-1)^n\,dx.$$ | Integrating by parts $(n-2)$ times

RECURRENCE RELATIONS

$$n\,P_n(x) = (2n-1)\,x\,P_{n-1}(x) - (n-1)\,P_{n-2}(x)$$

Or

$$(n+1)\,P_{n+1}(x) = (2n+1)\,x\,P_n(x) - n\,P_{n-1}(x)$$

We know that,

$$(1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x) \quad \ldots(1)$$

Differentiating both sides w.r.t. h, we get

$$-\frac{1}{2}(1-2xh+h^2)^{-3/2}(2h-2x) = \sum_{0}^{\infty} n\,h^{n-1} P_n(x)$$

$$\Rightarrow \quad (x-h)(1-2xh+h^2)^{-1/2} = (1-2xh+h^2)\sum_{0}^{\infty} n\,h^{n-1} P_n(x)$$

$$\Rightarrow \quad (x-h)\sum_{0}^{\infty} h^n P_n(x) = (1-2xh+h^2)\sum_{0}^{\infty} n\,h^{n-1} P_n(x)$$

Equating coefficient of h^{n-1} on both sides,

$$x\,P_{n-1}(x) - P_{n-2}(x) = n\,P_n(x) - 2x(n-1)\,P_{n-1}(x) + (n-2)\,P_{n-2}(x)$$

$$\Rightarrow \quad n\,P_n(x) = (2n-1)\,x\,P_{n-1}(x) - (n-1)\,P_{n-2}(x)$$

Replacing n by $(n+1)$, we get the other form.

$$n\,P_n(x) = x\,P_n'(x) - P'_{n-1}(x)$$

We know that, $$(1-2h\,x+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x) \quad \ldots(1)$$

Differentiating both sides of (1) w.r.t. h, we get

$$-\frac{1}{2}(1-2xh+h^2)^{-3/2}\,.\,(-2x+2h)=\sum_{0}^{\infty} n h^{n-1} P_n(x)$$

$$\Rightarrow \qquad (x-h)(1-2hx+h^2)^{-3/2}=\sum_{0}^{\infty} n h^{n-1} P_n(x) \qquad ...(2)$$

Differentiating both sides of (1) w.r.t. x, we get

$$-\frac{1}{2}(1-2hx+h^2)^{-3/2}\,.\,(-2h)=\sum_{0}^{\infty} h^{n} P_n'(x)$$

$$\Rightarrow \qquad (x-h)(1-2hx+h^2)^{-3/2}=(x-h)\sum_{0}^{\infty} h^{n-1} P_n'(x) \qquad ...(3)$$

Equating eqns. (2) and (3), we get

$$\sum_{0}^{\infty} n h^{n-1} P_n(x) = (x-h)\sum_{0}^{\infty} h^{n-1} P_n'(x)$$

Comparing the coefficient of h^{n-1} on both sides, we get

$$n P_n(x) = x P_n'(x) - P'_{n-1}(x)$$

$(2n + 1) P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$

From Recurrence relation (1),

$$(2n+1)\, x P_n(x) = (n+1) P_{n+1}(x) + n P_{n-1}(x)$$

Differentiating w.r.t. x, we get

$$(2n+1)[x P_n'(x) + P_n(x)] = (n+1) P'_{n+1}(x) + n P'_{n-1}(x) \qquad ...(1)$$

From Recurrence relation (2), $\quad x P_n'(x) = n P_n(x) + P'_{n-1}(x)$

From (1),

$$(2n+1)[n P_n(x) + P'_{n-1}(x) + P_n(x)] = (n+1) P'_{n+1}(x) + n P'_{n-1}(x)$$

$$\Rightarrow \qquad (2n+1)(n+1) P_n(x) = (n+1) P'_{n+1}(x) - (n+1) P'_{n-1}(x)$$

$$\Rightarrow \qquad (2n+1) P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$$

$(n + 1) P_n(x) = P'_{n+1}(x) - x P_n'(x)$

From Recurrence relation (3), we have

$$(2n+1) P_n(x) = P'_{n+1}(x) - P'_{n-1}(x) \qquad ...(1)$$

From Recurrence relation (2), we have

$$n P_n(x) = x P_n'(x) - P'_{n-1}(x) \qquad ...(2)$$

Subtraction yields, $\quad (n+1) P_n(x) = P'_{n+1}(x) - x P'_n(x)$

$(1 - x^2) P'_n(x) = n[P_{n-1}(x) - xP_n(x)]$

From Recurrence relation (4), we have

$$P_n'(x) - xP'_{n-1}(x) = nP_{n-1}(x) \qquad ...(1)$$

From Recurrence relation (2), we have

$$x P_n'(x) - P'_{n-1}(x) = n P_n(x) \qquad ...(2)$$

Multiplying (2) by x and subtracting from (1), we get

$$(1-x^2) P_n'(x) = n[P_{n-1}(x) - xP_n(x)]$$

NOTES

$$\boxed{(1 - x^2)\, P'_n(x) = (n + 1)\,[xP_n(x) - P_{n+1}(x)]}$$

Recurrence relation (1) may be written as

$$(\overline{n+1}+\overline{n})\, xP_n(x) = (n + 1)\, P_{n+1}(x) + nP_{n-1}(x)$$

$$\Rightarrow \quad (n + 1)\, xP_n(x) + nxP_n(x) = (n + 1)\, P_{n+1}(x) + nP_{n-1}(x)$$

or $$(n + 1)\,[xP_n(x) - P_{n+1}(x)] = n[P_{n-1}(x) - xP_n(x)]$$

$$= (1 - x^2)\, P_n'(x) \qquad | \text{ Using recurrence relation (5)}$$

$$\therefore \quad (1 - x^2)\, P_n'(x) = (n + 1)\,[xP_n(x) - P_{n+1}(x)].$$

BELTRAMI'S RESULT

$$\boxed{(2n + 1)(x^2 - 1)\, P_n' = n(n + 1)(P_{n+1} - P_{n-1})}$$

From Recurrence relation (5), we have

$$n(P_{n-1} - xP_n) = (1 - x^2)\, P_n' \qquad \ldots(1)$$

From Recurrence relation (6), we have

$$(n + 1)(xP_n - P_{n+1}) = (1 - x^2)\, P_n' \qquad \ldots(2)$$

From eqn. (1), $nP_{n-1} - nx\, P_n = (1 - x^2)\, P_n'$

$$\Rightarrow \quad xP_n = \frac{nP_{n-1} - (1 - x^2)\, P_n'}{n} \qquad \ldots(3)$$

From eqn. (2) $$xP_n - P_{n+1} = \frac{(1 - x^2)\, P_n'}{n + 1}$$

$$\Rightarrow \quad xP_n = P_{n+1} + \frac{(1 - x^2)\, P_n'}{n + 1} \qquad \ldots(4)$$

From (3) and (4),

$$\frac{nP_{n-1} - (1 - x^2) P_n'}{n} = P_{n+1} + \frac{(1 - x^2)\, P_n'}{n + 1}$$

$$= \frac{(n + 1)\, P_{n+1} + (1 - x^2) P_n'}{n + 1}$$

$$\Rightarrow \quad (n + 1)\,\{nP_{n-1} - (1 - x^2)\, P_n'\} = n\{(n + 1)\, P_{n+1} + (1 - x^2)\, P_n'\}$$

$$\Rightarrow \quad (2n + 1)(1 - x^2)\, P_n' = n\,(n + 1)\,\{P_{n-1} - P_{n+1}\}$$

$$\Rightarrow \quad (2n + 1)(x^2 - 1)\, P_n' = n(n + 1)\,\{P_{n+1} - P_{n-1}\}.$$

ORTHOGONALITY OF LEGENDRE POLYNOMIALS

We shall show that

$$\boxed{\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = \begin{cases} 0, & \text{if } m \neq n \\ \dfrac{2}{2n + 1}, & \text{if } m = n \end{cases}}$$

Case I. *When* $m \neq n$

We know that $P_m(x)$ and $P_n(x)$ are the solutions of the equations

$$(1 - x^2)u'' - 2xu' + m(m + 1)u = 0 \quad ...(1)$$

and $$(1 - x^2)v'' - 2xv' + n(n + 1)v = 0 \quad ...(2)$$

Multiplying (1) by v and (2) by u and subtracting, we get

$$(1 - x^2)(u''v - v''u) - 2x(u'v - v'u) + [m(m + 1) - n(n + 1)]\, uv = 0$$

or $$\frac{d}{dx}[(1 - x^2)(u'v - v'u)] + (m - n)(m + n + 1)uv = 0$$

or $$(n - m)(n + m + 1)\, uv = \frac{d}{dx}[(1 - x^2)(u'v - v'u)]$$

Integrating w.r.t. x from -1 to 1, we get

$$(n - m)(n + m + 1)\int_{-1}^{1} uv\, dx = \Big[(1 - x^2)(u'v - v'u)\Big]_{-1}^{1} = 0$$

Hence $\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = 0$, since $m \neq n$.

Case II. *When* $m = n$

We know that $$(1 - 2xh + h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x)$$

Squaring both sides, we get

$$(1 - 2xh + h^2)^{-1} = \sum_{n=0}^{\infty} [h^n P_n(x)]^2 = \sum_{n=0}^{\infty} h^{2n}[P_n(x)]^2 + 2\sum_{m=0}^{\infty}\sum_{\substack{n=0\\(m\neq n)}}^{\infty} h^{m+n} P_m(x)\, P_n(x)$$

Integrating w.r.t. x between the limits -1 to 1, we have

$$\sum_{n=0}^{\infty}\int_{-1}^{1} h^{2n}[P_n(x)]^2\, dx + 2\sum_{m=0}^{\infty}\sum_{\substack{n=0\\(m\neq n)}}^{\infty}\int_{-1}^{1} h^{m+n} P_m(x)\, P_n(x)\, dx = \int_{-1}^{1}\frac{dx}{1 - 2xh + h^2}$$

or $$\sum_{n=0}^{\infty}\int_{-1}^{1} h^{2n}[P_n(x)]^2\, dx = \int_{-1}^{1}\frac{dx}{1 - 2xh + h^2}$$

| Since other integrals on the LHS vanish by Case I as $m \neq n$

$$= -\frac{1}{2h}\Big[\log(1 - 2xh + h^2)\Big]_{-1}^{1} = -\frac{1}{2h}[\log(1 - h)^2 - \log(1 + h)^2]$$

$$= \frac{1}{h}[\log(1 + h) - \log(1 - h)]$$

$$= \frac{1}{h}\left[\left(h - \frac{h^2}{2} + \frac{h^3}{3} - \frac{h^4}{4} + ...\right) + \left(h + \frac{h^2}{2} + \frac{h^3}{3} + \frac{h^4}{4} + ...\right)\right]$$

$$= \frac{2}{h}\left[h + \frac{h^3}{3} + \frac{h^5}{5} + ...\right]$$

or $$\sum_{n=0}^{\infty} h^{2n}\int_{-1}^{1}[P_n(x)]^2\, dx = 2\left(1 + \frac{h^2}{3} + \frac{h^4}{5} + ... + \frac{h^{2n}}{2n + 1} + ...\right)$$

NOTES

Equating the coefficients of h^{2n} on the two sides, we get

$$\int_{-1}^{1} [P_n(x)]^2 \, dx = \frac{2}{2n+1}.$$

LAPLACE'S INTEGRAL OF FIRST KIND

$$\mathbf{P_n(x) = \frac{1}{\pi}\int_0^{\pi} \{x \pm \sqrt{x^2 - 1}\cos\phi\}^n \, d\phi}$$

We know that,

$$\int_0^{\pi} \frac{d\phi}{a \pm b\cos\phi} = \frac{\pi}{\sqrt{a^2 - b^2}}; \; a > b \qquad \ldots(1)$$

Replace a by $(1 - xz)$ and b by $z\sqrt{x^2 - 1}$

$\therefore \qquad a^2 - b^2 = (1 - xz)^2 - z^2(x^2 - 1) = 1 - 2xz + z^2$

Then (1) becomes

$$\int_0^{\pi} \frac{d\phi}{1 - xz \pm z\sqrt{x^2 - 1}\cos\phi} = \frac{\pi}{\sqrt{1 - 2xz + z^2}}$$

$$\Rightarrow \qquad \int_0^{\pi} \frac{d\phi}{1 - z\{x \mp \sqrt{x^2 - 1}\cos\phi\}} = \pi(1 - 2xz + z^2)^{-1/2}$$

Let $z\{x \mp \sqrt{x^2 - 1}\cos\phi\} = t$, then

$$\int_0^{\pi} \frac{d\phi}{1 - t} = \pi \sum_{n=0}^{\infty} z^n P_n(x) \qquad \ldots(2)$$

If $|\, t \,| < 1$, then $\qquad (1 - t)^{-1} = \sum_{n=0}^{\infty} t^n$

$\therefore$ From (2), $\quad \int_0^{\pi} \sum_{n=0}^{\infty} [z^n \{x \mp \sqrt{x^2 - 1}\cos\phi\}^n]\, d\phi = \pi \sum_{n=0}^{\infty} z^n P_n(x)$

Equating the coefficient of z^n on both sides, we get

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi} \{x \pm \sqrt{x^2 - 1}\cos\phi\}^n \, d\phi.$$

LAPLACE'S INTEGRAL OF SECOND KIND

$$\mathbf{P_n(x) = \frac{1}{\pi}\int_0^{\pi} \frac{d\phi}{\{x \pm \sqrt{x^2 - 1}\cos\phi\}^{n+1}}}.$$

We know that

$$\int_0^{\pi} \frac{d\phi}{a \pm b\cos\phi} = \frac{\pi}{\sqrt{a^2 - b^2}} \; ; a > b \qquad \text{...(1)}$$

Put $a = xz - 1$ and $b = z\sqrt{x^2 - 1}$ so that

$$a^2 - b^2 = (xz - 1)^2 - z^2(x^2 - 1) = 1 - 2xz + z^2$$

With above substitutions, (1) becomes,

$$\int_0^{\pi} \frac{d\phi}{xz - 1 \pm z\sqrt{x^2 - 1}\cos\phi} = \frac{\pi}{\sqrt{1 - 2xz + z^2}}$$

$$= \frac{\pi}{z\sqrt{1 - 2\left(\frac{1}{z}\right)x + \left(\frac{1}{z}\right)^2}} = \frac{\pi}{z}\sum_{n=0}^{\infty} \frac{1}{z^n} P_n(x)$$

$$\therefore \quad \int_0^{\pi} \frac{d\phi}{z\{x \pm \sqrt{x^2 - 1}\cos\phi\} - 1} = \frac{\pi}{z^{n+1}} \sum_{n=0}^{\infty} P_n(x)$$

$$\Rightarrow \quad \int_0^{\pi} \frac{d\phi}{t - 1} = \frac{\pi}{z^{n+1}} \sum_{n=0}^{\infty} P_n(x) \qquad \text{...(2)} \quad \text{where } t = z\{x \pm \sqrt{x^2 - 1}\cos\phi\}$$

Now, $\qquad \text{LHS} = \int_0^{\pi} \frac{d\phi}{t\left(1 - \frac{1}{t}\right)} = \int_0^{\pi} \frac{1}{t}\left(1 - \frac{1}{t}\right)^{-1} d\phi$

If $\left|\frac{1}{t}\right| < 1$, then $\quad \text{LHS} = \int_0^{\pi} \frac{1}{t}\left(1 + \frac{1}{t} + \frac{1}{t^2} + \cdots\cdots + \frac{1}{t^n} + \cdots\cdots\right) d\phi = \int_0^{\pi} \sum_{n=0}^{\infty} \frac{1}{t^{n+1}} d\phi$

$$= \int_0^{\pi} \sum_{n=0}^{\infty} \frac{1}{z^{n+1}\{x \pm \sqrt{x^2 - 1}\cos\phi\}^{n+1}} d\phi$$

Now, comparing and equating the coefficients of $\frac{1}{z^{n+1}}$ on both sides of eqn.(2), we get

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi} \frac{d\phi}{\{x \pm \sqrt{x^2 - 1}\cos\phi\}^{n+1}}.$$

CRISTOFFEL'S EXPANSION FORMULA

$P_n'(x) = (2n - 1)\, P_{n-1}(x) + (2n - 5)\, P_{n-3}(x) + (2n - 9)\, P_{n-5}(x) + \cdots\cdots +$ Last term.

where $\quad$ Last term = $\begin{Bmatrix} 3P_1\,; & \text{when } n \text{ is even} \\ P_0\,; & \text{when } n \text{ is odd} \end{Bmatrix}$.

From Recurrence relation (3), we know that,

$$(2n + 1)\, P_n = P'_{n+1} - P'_{n-1}$$

$$\Rightarrow \qquad P'_{n+1} = (2n + 1)\, P_n + P'_{n-1} \qquad \text{...(1)}$$

Replace n by $(n-1)$, then

$$P_n' = (2n-1)\,P_{n-1} + P'_{n-2} \qquad ...(2)$$

Now replace n by $(n-2)$ in (2), we get

$$P'_{n-2} = (2n-5)\,P_{n-3} + P'_{n-4} \qquad ...(3)$$

$\therefore$ From (2), $\quad P_n' = (2n-1)\,P_{n-1} + (2n-5)\,P_{n-3} + P'_{n-4}$

Proceeding in this manner, for the last term, two cases arise:

Case I. *When n is even:*

$$P_2' = 3P_1 + P_0' = 3P_1 \qquad \Big|\because\ P_0 = 1 \text{ and } P_0' = 0$$

so, $\quad$ last term $= 3P_1$

Case II. *When n is odd:*

$$P_3' = 5P_2 + P_1' \qquad \Big|\because\ P_1 = x\ \therefore\ P_1' = 1 = P_0$$

$$= 5P_2 + P_0$$

so, $\quad$ last term $= P_0$.

CRISTOFFEL'S SUMMATION FORMULA

The sum of first $(n+1)$ terms of the series

$$\sum_{m=0}^{\infty} (2m+1)\,P_m(x)\,P_m(y) = \frac{(n+1)\,[P_{n+1}(x)\,P_n(y) - P_n(x)\,P_{n+1}(y)]}{x-y}$$

By Recurrence relation (1),

$$(2m+1)\,xP_m(x) = (m+1)\,P_{m+1}(x) + mP_{m-1}(x) \qquad ...(1)$$

Similarly, $\quad (2m+1)\,yP_m(y) = (m+1)\,P_{m+1}(y) + mP_{m-1}(y) \qquad ...(2)$

Multiplying (1) by $P_m(y)$ and (2) by $P_m(x)$ and then subtracting (2) from (1), we get

$$\begin{aligned}(2m+1)\,(x-y)\,P_m(x)\,P_m(y) &= (m+1)\,P_{m+1}(x)\,P_m(y) + mP_{m-1}(x)\,P_m(y)\\ &\quad - (m+1)\,P_{m+1}(y)\,P_m(x) - mP_{m-1}(y)\,P_m(x)\\ &= (m+1)\,[P_{m+1}(x)\,P_m(y) - P_{m+1}(y)\,P_m(x)]\\ &\quad - m\,[P_{m-1}(y)\,P_m(x) - P_{m-1}(x)\,P_m(y)]\end{aligned}$$

Put 0, 1, 2, 3, ..., n for m in succession, we get

$$(x-y)\,P_0(x)\,P_0(y) = P_1(x)\,P_0(y) - P_1(y)\,P_0(x)$$

$$3(x-y)\,P_1(x)\,P_1(y) = 2\{P_2(x)\,P_1(y) - P_2(y)\,P_1(x)\} - \{P_0(y)\,P_1(x) - P_0(x)\,P_1(y)\}$$

$$5(x-y)\,P_2(x)\,P_2(y) = 3\,\{P_3(x)\,P_2(y) - P_3(y)\,P_2(x)\} - 2\{P_1(y)\,P_2(x) - P_1(x)\,P_2(y)\}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\begin{aligned}(2n+1)\,(x-y)\,P_n(x)\,P_n(y) &= (n+1)\,[P_{n+1}(x)\,P_n(y) - P_{n+1}(y)\,P_n(x)]\\ &\quad - n\,[P_{n-1}(y)\,P_n(x) - P_{n-1}(x)\,P_n(y)]\end{aligned}$$

Adding simultaneously, we get

$$(x-y)\,[P_0(x)\,P_0(y) + 3P_1(x)\,P_1(y) + 5P_2(x)\,P_2(y) + \cdots + (2n+1)\,P_n(x)\,P_n(y)]$$

$$= (n+1)\,[P_{n+1}(x)\,P_n(y) - P_{n+1}(y)\,P_n(x)]$$

$$\Rightarrow \quad P_0(x)\, P_0(y) + 3P_1(x)\, P_1(y) + \cdots + (2n+1)\, P_n(x)\, P_n(y)$$
$$= \left(\frac{n+1}{x-y}\right) \{P_{n+1}(x)\, P_n(y) - P_{n+1}(y)\, P_n(x)\}$$

$\therefore$ Sum of first n terms of the series

$$\sum_{m=0}^{\infty} (2m+1)\, P_m(x)\, P_m(y) = \frac{(n+1)\{P_{n+1}(x)\, P_n(y) - P_{n+1}(y)\, P_n(x)\}}{x-y}.$$

EXPANSION OF A FUNCTION IN A SERIES OF LEGENDRE POLYNOMIALS (FOURIER-LEGENDRE SERIES)

The orthogonal property of Legendre polynomials enables us to expand a function $f(x)$, defined from $x = -1$ to $x = 1$ in a series of Legendre polynomials.

Let
$$f(x) = \sum_{n=0}^{\infty} a_n\, P_n(x) = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x) + \ldots \qquad \ldots(1)$$

To determine a_n, multiplying both sides of (1) by $P_n(x)$ and integrating w.r.t. x from -1 to 1, we have

$$\int_{-1}^{1} f(x)\, P_n(x)\, dx = a_n \int_{-1}^{1} P_n^{\,2}(x)\, dx = a_n \left(\frac{2}{2n+1}\right)$$

$$\Rightarrow \quad \boxed{a_n = \left(n + \frac{1}{2}\right) \int_{-1}^{1} f(x)\, P_n(x)\, dx}$$

Expansion of $f(x)$ given by (1) is known as *Fourier-Legendre series.*

SOLVED EXAMPLES

Example 13. *Prove that:* $\int_{-1}^{1} (1 - x^2)\, P_m'\, P_n'\, dx = 0$

where m and n are distinct positive integers and $m \neq n$.

Sol. $\int_{-1}^{1} (1 - x^2)\, P_m'\, P_n'\, dx$

$$= \Big[(1 - x^2)\, P_m'\, P_n\Big]_{-1}^{1} - \int_{-1}^{1} P_n \left[\frac{d}{dx}\{(1 - x^2)\, P_m'\}\right] dx$$

| Integrating by parts

$$= -\int_{-1}^{1} P_n \frac{d}{dx} \{(1 - x^2)\, P_m'\}\, dx$$

$$= -\int_{-1}^{1} P_n\{-m(m+1)\, P_m\}\, dx$$ | From Legendre's differential equation

$$= m(m+1) \int_{-1}^{1} P_n\, P_m\, dx = m(m+1)\,.\,0 = 0$$

NOTES

Example 14. *Prove that:* $\int_{-1}^{1} x^2\, P_{n-1}(x)\, P_{n+1}(x)\, dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$.

Sol. Recurrence relation (1) is

$$(n + 1)\, P_{n+1} = (2n + 1)\, xP_n - n\, P_{n-1}$$

$$\Rightarrow \qquad (2n + 1)\, xP_n = (n + 1)\, P_{n+1} + nP_{n-1} \qquad \ldots(1)$$

Replacing n by $n + 1$ and $n - 1$ respectively in (1), we get

$$(2n + 3)\, xP_{n+1} = (n + 2)\, P_{n+2} + (n + 1)\, P_n \qquad \ldots(2)$$

and

$$(2n - 1)\, xP_{n-1} = nP_n + (n - 1)\, P_{n-2} \qquad \ldots(3)$$

Multiplying (2) and (3) and integrating within limits – 1 and 1, we get

$$(2n + 3)\,(2n - 1) \int_{-1}^{1} x^2 P_{n+1} P_{n-1}\, dx$$

$$= n\,(n + 1) \int_{-1}^{1} P_n^{\,2} dx + n(n + 2) \int_{-1}^{1} P_n\, P_{n+2} dx + (n^2 - 1) \int_{-1}^{1} P_{n-2}\, P_n\, dx$$

$$+ (n + 2)\,(n - 1) \int_{-1}^{1} P_{n-2}\, P_{n+2}\, dx$$

$$= n(n + 1)\,.\, \frac{2}{2n+1} \qquad \text{[Using orthogonal properties]}$$

$$\therefore \quad \int_{-1}^{1} x^2\, P_{n+1}\, P_{n-1}\, dx = \frac{2n(n+1)}{(2n+1)(2n-1)(2n+3)}.$$

Example 15. *Prove that:* $\int_{0}^{1} P_n^2(x)\, dx = \frac{1}{2n+1}$.

Sol. We know that by orthogonal property, $\int_{-1}^{1} P_n^2(x)\, dx = \frac{2}{2n+1}$

$$\Rightarrow \qquad \int_{-1}^{0} P_n^2(x)\, dx + \int_{0}^{1} P_n^2(x)\, dx = \frac{2}{2n+1}$$

Put $x = -y$ in first integral, then $dx = -dy$

$$-\int_{1}^{0} P_n^2(-y)\, dy + \int_{0}^{1} P_n^2(x)\, dx = \frac{2}{2n+1}$$

$$\Rightarrow \qquad \int_{0}^{1} P_n^2(-x)\, dx + \int_{0}^{1} P_n^2(x)\, dx = \frac{2}{2n+1}$$

$$\Rightarrow \qquad \int_{0}^{1} (-1)^{2n}\, P_n^2(x) dx + \int_{0}^{1} P_n^2(x)\, dx = \frac{2}{2n+1}$$

$$\Rightarrow \qquad 2\int_{0}^{1} P_n^2(x)\, dx = \frac{2}{2n+1}$$

$$\Rightarrow \qquad \int_{0}^{1} P_n^2(x)\, dx = \frac{1}{2n+1}.$$

Example 16. *Prove that:*

$$\int_{-1}^{1} P_n(x)\,(1 - 2xt + t^2)^{-1/2}\, dx = \frac{2t^n}{2n+1} \quad \textit{where n is a positive integer.}$$

Sol. $\int_{-1}^{1} P_n(x)(1-2xt+t^2)^{-1/2}\,dx$

$$= \int_{-1}^{1} P_n(x)\{\Sigma t^n P_n(x)\}\,dx$$

$$= t^n \int_{-1}^{1} P_n^2(x)\,dx \qquad \Big|\ \text{All other terms vanish since } \int_{-1}^{1} P_m(x)P_n(x)\,dx = 0;\ m \neq n$$

$$= t^n \cdot \frac{2}{2n+1}. \qquad \Big|\ \text{By II orthogonal property}$$

Example 17. *Prove that:* $\int_{-1}^{1} x^m P_n(x)\,dx = 0$, *if* $m < n$.

Sol. $\int_{-1}^{1} x^m P_n(x)\,dx = \int_{-1}^{1} x^m \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n\,dx$ (Using Rodrigue's Formula)

$$= \frac{1}{2^n n!}\int_{-1}^{1} x^m \frac{d^n}{dx^n}(x^2-1)^n\,dx$$

Integrating by parts, we get

$$= \frac{1}{2^n n!}\left[\left\{x^m \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\right\}_{-1}^{1} - \int_{-1}^{1} mx^{m-1}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\,dx\right]$$

$$= 0 - \frac{m}{2^n n!}\int_{-1}^{1} x^{m-1}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\,dx$$

Similarly, $\int_{-1}^{1} x^m P_n(x)\,dx = (-1)^2 \frac{m(m-1)}{2^n n!}\int_{-1}^{1} x^{m-2}\frac{d^{n-2}}{dx^{n-2}}(x^2-1)^n\,dx$

Integrating $(m-2)$ times in all, we get

$$I = (-1)^m \frac{m(m-1)\ldots 1}{2^n n!}\int_{-1}^{1}\frac{d^{n-m}}{dx^{n-m}}(x^2-1)^n\,dx$$

$$= \frac{(-1)^m m!}{2^n n!}\int_{-1}^{1}\frac{d^{n-m}}{dx^{n-m}}(x^2-1)^n\,dx$$

$$= \frac{(-1)^m m!}{2^n n!}\left[\frac{d^{n-m-1}}{dx^{n-m-1}}(x^2-1)^n\right]_{-1}^{1} = 0.$$

Example 18. *Prove that:* $\frac{P_{n+1}-P_{n-1}}{2n+1} = \int P_n\,dx + c$.

Sol. From Recurrence relation (3), we have

$$P'_{n+1} - P'_{n-1} = (2n+1)P_n$$

Integrating both sides w.r.t. x, we get

$$\frac{P_{n+1}-P_{n-1}}{2n+1} = \int P_n\,dx + c.$$

Example 19. *Prove that:* $xP_n' = nP_n + (2n-3)P_{n-2} + (2n-7)P_{n-4} + \cdots$

Sol. From Recurrence relation (2),

$$xP_n' = nP_n + P'_{n-1} \qquad \ldots(1)$$

NOTES

From Recurrence relation (3),

$$P'_{n+1} = (2n + 1)\ P_n + P'_{n-1} \qquad ...(2)$$

Replacing n by $n - 2$ in (2), we get

$$P'_{n-1} = (2n - 3)\ P_n + P'_{n-3} \qquad ...(3)$$

Replacing n by $n - 4$ in (2), we get

$$P'_{n-3} = (2n - 7)\ P_{n-4} + P'_{n-5} \qquad ...(4)$$

$\therefore$ From (1), (3) and (4),

$$xP'_n = nP_n + (2n - 3)\ P_n + (2n - 7)\ P_{n-4} + \cdots$$

Example 20. *Prove that:* $\int_{-1}^{1} xP_n P'_n\, dx = \dfrac{2n}{2n+1}$.

Sol.

$$\int_{-1}^{1} P_n (x P'_n)\, dx = \int_{-1}^{1} P_n [nP_n + (2n-3)\, P_{n-2} + (2n-7)\, P_{n-4} + \cdots]\, dx$$

$$= \int_{-1}^{1} nP_n^2\, dx + (2n-3)\int_{-1}^{1} P_n P_{n-2}\, dx + (2n-7)\int_{-1}^{1} P_n P_{n-4}\, dx + \cdots$$

$$= n \cdot \frac{2}{2n+1} + 0 + 0 + \cdots \qquad | \text{ Using orthogonal property}$$

$$= \frac{2n}{2n+1}$$

Example 21. *Prove that:* $\int_{-1}^{1} (x^2 - 1)\, P_{n+1}\, P'_n\, dx = \dfrac{2n(n+1)}{(2n+1)(2n+3)}$.

Sol. From Recurrence relation (5),

$$n(P_{n-1} - xP_n) = (1 - x^2)\ P'_n$$

$$\Rightarrow \qquad (x^2 - 1)\ P'_n = n(xP_n - P_{n-1}) \qquad ...(1)$$

Now, $\int_{-1}^{1} (x^2 - 1)\, P_{n+1} P'_n\, dx = \int_{-1}^{1} \{(x^2 - 1) P'_n\}\, P_{n+1}\, dx$

$$= \int_{-1}^{1} n\,(xP_n - P_{n-1})\, P_{n+1}\, dx = n\int_{-1}^{1} xP_n\, P_{n+1}\, dx - n\int_{-1}^{1} P_{n-1}\, P_{n+1}\, dx$$

$$= n\int_{-1}^{1} xP_n\, P_{n+1}\, dx \qquad ...(2) \qquad \Big|\ \because \int_{-1}^{1} P_{n-1}\, P_{n+1}\, dx = 0$$

From Recurrence relation (1),

$$(2n + 1)\ xP_n = (n + 1)\ P_{n+1} + nP_{n-1}$$

$$\Rightarrow \qquad xP_n = \frac{(n+1)\, P_{n+1} + nP_{n-1}}{(2n+1)}$$

$\therefore$ From (2), $\int_{-1}^{1} (x^2 - 1) P_{n+1}\, P'_n\, dx = n \int_{-1}^{1} \left[\dfrac{(n+1)\, P_{n+1} + n\, P_{n-1}}{2n+1}\right] . P_{n+1}\, dx$

$$= \frac{n(n+1)}{2n+1}\int_{-1}^{1} P'_{n+1}\, dx + \frac{n^2}{2n+1}\int_{-1}^{1} P_{n-1}\, P_{n+1}\, dx$$

$$= \frac{n(n+1)}{(2n+1)} \cdot \frac{2}{2(n+1)+1} + 0 = \frac{2n(n+1)}{(2n+1)(2n+3)}.$$

Example 22. *Prove that:* $\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} P_n(x) = \frac{1}{2} \log\left(\frac{1+x}{1-x}\right)$.

Sol. We know that

$$\sum_{n=0}^{\infty} h^n P_n(x) = (1 - 2xh + h^2)^{-1/2}$$

Integrating both sides w.r.t. h from 0 to h, we get

$$\sum_{n=0}^{\infty} \frac{h^{n+1}}{n+1} P_n(x) = \int_0^h \frac{dh}{\sqrt{1-2hx+h^2}} = \int_0^h \frac{dh}{\sqrt{(h-x)^2+(1-x^2)}}; \text{ if } |x| < 1$$

$$= \log \frac{(h-x)+\sqrt{h^2-2hx+1}}{1-x}$$

Putting $h = x$ in the expression, we get

$$\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} P_n(x) = \log\left(\frac{\sqrt{1-x^2}}{1-x}\right) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right).$$

Example 23. *If* $f(x) = \left.\begin{matrix} 0, & -1 < x < 0 \\ x, & 0 < x < 1 \end{matrix}\right\}$ *show that:*

$$f(x) = \frac{1}{4} P_0(x) + \frac{1}{2} P_1(x) + \frac{5}{16} P_2(x) - \frac{3}{32} P_4(x) + \dots$$

Sol. We know that

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x) \quad \dots(1) \quad \text{where } a_n = \left(n+\frac{1}{2}\right)\int_{-1}^{1} f(x) P_n(x)\, dx$$

$$= \left(\frac{2n+1}{2}\right)\left[\int_{-1}^{0} f(x) P_n(x)\, dx + \int_0^1 f(x) P_n(x)\, dx\right]$$

$$= \left(\frac{2n+1}{2}\right)\int_0^1 f(x) P_n(x)\, dx \quad \dots(2)$$

Putting $n = 0, 1, 2, 3, 4, \dots$ successively in (2), we get

$$a_0 = \frac{1}{2}\int_0^1 x P_0(x)\, dx = \frac{1}{2}\left(\frac{x^2}{2}\right)_0^1 = \frac{1}{4}$$

$$a_1 = \frac{3}{2}\int_0^1 x P_1(x)\, dx = \frac{3}{2}\left(\frac{x^3}{3}\right)_0^1 = \frac{1}{2}$$

$$a_2 = \frac{5}{2}\int_0^1 x P_2(x)\, dx = \frac{5}{2}\int_0^1 x\left(\frac{3x^2-1}{2}\right) dx = \frac{5}{16}$$

$$a_3 = \frac{7}{2}\int_0^1 x P_3(x)\, dx = \frac{7}{2}\int_0^1 x\left(\frac{5x^3-3x}{2}\right) dx = 0$$

$$a_4 = \frac{9}{2}\int_0^1 x P_4(x)\, dx = \frac{9}{2}\int_0^1 x\left(\frac{35x^4-30x^2+3}{8}\right) dx = \frac{-3}{32} \text{ and so on.}$$

Putting these values in (1), we get

$$f(x) = \frac{1}{4} P_0(x) + \frac{1}{2} P_1(x) + \frac{5}{16} P_2(x) - \frac{3}{32} P_4(x) + \cdots$$

NOTES

Example 24. *Compute the first three non-vanishing terms in the Fourier-Legendre series over the interval (– 1, 1) of the function*

$$f(x) = \begin{cases} \dfrac{1}{2\varepsilon}, & |x| < \varepsilon \\ 0 \;, & \varepsilon < |x| < 1 \end{cases}.$$

Sol. Let the Fourier-Legendre series be

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x) \quad \ldots(1)$$

where, $$a_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)\, P_n(x)\, dx = \frac{2n+1}{2}\int_{-\varepsilon}^{\varepsilon} f(x)\, P_n(x)\, dx \quad \ldots(2)$$

Putting $n = 0, 1, 2, 3, \ldots$ successively in (2), we get

$$a_0 = \frac{1}{2}\int_{-\varepsilon}^{\varepsilon} \frac{1}{2\varepsilon} P_0(x)\, dx = \frac{1}{4\varepsilon}(2\varepsilon) = \frac{1}{2} \qquad | \because \; P_0(x) = 1$$

$$a_1 = \frac{3}{2}\int_{-\varepsilon}^{\varepsilon} \frac{1}{2\varepsilon} P_1(x)\, dx = \frac{3}{4\varepsilon}\int_{-\varepsilon}^{\varepsilon} x\, dx = 0$$

Therefore $a_n = 0$ for all n odd.

Now, $$a_2 = \frac{5}{2}\int_{-\varepsilon}^{\varepsilon} \frac{1}{2\varepsilon} P_2(x)\, dx = \frac{5}{4\varepsilon}\int_{-\varepsilon}^{\varepsilon}\left(\frac{3x^2-1}{2}\right) dx = \frac{5}{8\varepsilon}\left(x^3 - x\right)_{-\varepsilon}^{\varepsilon}$$

$$\Rightarrow \qquad a_2 = \frac{5}{4\varepsilon}(\varepsilon^3 - \varepsilon) = \frac{5}{4}(\varepsilon^2 - 1)$$

Also, $$a_4 = \frac{9}{2}\int_{-\varepsilon}^{\varepsilon} \frac{1}{2\varepsilon} P_4(x)\, dx$$

$$= \frac{9}{4\varepsilon}\int_{-\varepsilon}^{\varepsilon}\left(\frac{35x^4 - 30x^2 + 3}{8}\right) dx = \frac{9}{32\varepsilon}\left(7x^5 - 10x^3 + 3x\right)_{-\varepsilon}^{\varepsilon}$$

$$\Rightarrow \qquad a_4 = \frac{9}{16}(7\varepsilon^5 - 10\varepsilon^2 + 3)$$

and so on ......

Hence the Fourier-Legendre series is

$$f(x) = \frac{1}{2}P_0(x) + \frac{5}{4}(\varepsilon^2 - 1)\, P_2(x) + \frac{9}{16}(7\varepsilon^5 - 10\varepsilon^2 + 3)\, P_4(x) + \cdots$$

Example 25. *Prove that:* $\displaystyle\sum_{r=0}^{n} (2r+1)\, P_r = P'_n + P'_{n+1}$.

Sol. From Recurrence relation (3), we have

$$(2n+1)\, P_n = P'_{n+1} - P'_{n-1}$$

Putting $n = 1, 2, 3, \ldots, n$, we get

$$3P_1 = P_2' - P_0'$$
$$5P_2 = P_3' - P_1'$$
$$7P_3 = P_4' - P_2'$$
$$\vdots \qquad \vdots$$
$$(2n-1)\, P_{n-1} = P_n' - P'_{n-2}$$
$$(2n+1)\, P_n = P'_{n+1} - P'_{n-1}$$

NOTES

Adding simultaneously, we get

$$3P_1 + 5P_2 + 7P_3 + \cdots + (2n+1)\,P_n = P'_{n+1} + P'_n - P'_0 - P'_1$$
$$= P'_{n+1} + P'_n - 1 = P'_{n+1} + P'_n - P_0$$

$\Rightarrow \quad P_0 + 3P_1 + 5P_2 + 7P_3 + \cdots + (2n+1)\,P_n = P'_{n+1} + P'_n$

$$\Rightarrow \quad \sum_{r=0}^{n} (2r+1)\,P_r = P'_{n+1} + P'_n.$$

Example 26. *If $P_n(x)$ is a Legendre polynomial of degree n and α is such that $P_n(\alpha) = 0$. Show that $P_{n-1}(\alpha)$ and $P_{n+1}(\alpha)$ are of opposite signs.*

Sol. From Recurrence relation (1), we have

$$(2n+1)\,xP_n(x) = (n+1)\,P_{n+1}(x) + nP_{n-1}(x) \qquad \ldots(1)$$

Given that $P_n(\alpha) = 0$ ...(2)

$\therefore$ Put $x = \alpha$ in (1) and using (2), we get

$$(2n+1)\,\alpha\,.\,0 = (n+1)\,P_{n+1}(\alpha) + nP_{n-1}(\alpha)$$

$$\Rightarrow \quad \frac{P_{n+1}(\alpha)}{P_{n-1}(\alpha)} = -\frac{n}{n+1} \qquad \ldots(3)$$

Since n is a positive integer so RHS of (3) is negative. Hence (3) shows that $P_{n+1}(\alpha)$ and $P_{n-1}(\alpha)$ are of opposite signs.

Example 27. *Show that all the roots of $P_n(x) = 0$ are real and lie between -1 and 1.*

Sol. Let $$f(x) = (x^2-1)^n = (x-1)^n\,(x+1)^n \qquad \ldots(1)$$

From (1), we see that $f(x)$ vanishes for $x = 1$ and $x = -1$ hence by Rolle's theorem, $f'(x)$ must vanish at least once for some value α of x lying between -1 and 1.

From (1), we have

$$f'(x) = n(x-1)^{n-1}\,(x+1)^n + n(x-1)^n\,(x+1)^{n-1}$$

which shows that $f'(x)$ vanishes at $x = 1$ and $x = -1$.

Again, we have already shown that $f'(x)$ vanishes at $x = \alpha$. Now applying Rolle's theorem to $f'(x)$ two times, we conclude that $f''(x)$ must vanish at $x = \beta$ between α and 1.

Proceeding in this manner, we conclude that $f^{(n)}(x) = 0$ must have n real roots lying between -1 and 1.

By Rodrigue's formula and eqn. (1), we have

$$P_n(x) = \frac{1}{2^n\,n!}\,f^{(n)}(x)$$

Hence we see that $P_n(x) = 0$ must have n real roots lying between -1 and 1.

Example 28. *Prove that:* $\int_x^1 P_n(x)\,dx = (P_{n-1} - P_{n+1})/(2n+1)$.

Sol. From Recurrence relation (3), we have

$$(2n+1)\,P_n(x) = P'_{n+1}(x) - P'_{n+1}(x)$$

$$\Rightarrow \quad P_n(x) = \frac{1}{2n+1}\frac{d}{dx}\,[P_{n+1}(x) - P_{n-1}(x)]$$

Integrating w.r.t. x between limits x to 1, we get

$$\int_x^1 P_n(x)\,dx = \frac{1}{2n+1}\Big[P_{n+1}(x) - P_{n-1}(x)\Big]_x^1$$

$$= \frac{1}{2n+1}\,[\{P_{n+1}(1) - P_{n-1}(1)\} - \{P_{n+1}(x) - P_{n-1}(x)\}]$$

$$= \frac{1}{2n+1}\,\{P_{n-1}(x) - P_{n+1}(x)\}. \qquad \Big|\ \because\ P_{n+1}(1) = 1 \text{ and } P_{n-1}(1) = 1$$

Example 29. *Prove that:* $\int_{-1}^{1} x^2 P_n^2\,dx = \frac{1}{8(2n-1)} + \frac{3}{4(2n+1)} + \frac{1}{8(2n+3)}$.

Sol. From Recurrence relation (1),

$$(2n+1)xP_n = (n+1)\,P_{n+1} + nP_{n-1}$$

Squaring both sides, we get

$$(2n+1)^2\,x^2P_n^{\ 2} = (n+1)^2\,P^2_{n+1} + n^2\,P^2_{n-1} + 2n(n+1)\,P_{n-1}P_{n+1}$$

Integrating w.r.t. x between limits – 1 to 1, we get

$$(2n+1)^2\int_{-1}^{1} x^2\,P_n^{\ 2}\,dx = (n+1)^2\int_{-1}^{1} P^2_{n+1}\,dx + n^2\int_{-1}^{1} P^2_{n-1}\,dx + 2n\,(n+1)\int_{-1}^{1} P_{n-1}P_{n+1}\,dx$$

$$= (n+1)^2\,.\,\frac{2}{[2(n+1)+1]} + n^2\,.\,\frac{2}{[2(n-1)+1]} + 0$$

$$\Rightarrow \int_{-1}^{1} x^2\,P_n^2\,dx = \frac{2}{(2n+1)^2}\left[\frac{(n+1)^2}{2n+3} + \frac{n^2}{2n-1}\right]$$

$$= \frac{1}{8(2n-1)} + \frac{3}{4(2n+1)} + \frac{1}{8(2n+3)}$$

(on resolving into partial fractions)

Example 30. *Show that:* $\int_{-1}^{1} xP_n(x)\,P_{n-1}(x)\,dx = \frac{2n}{4n^2-1}$.

Sol. Recurrence relation (1) is

$$nP_n = (2n-1)\,x\,P_{n-1} - (n-1)\,P_{n-2} \qquad \ldots(1)$$

From (1), $$x\,P_{n-1} = \frac{1}{2n-1}\,[nP_n + (n-1)\,P_{n-2}]$$

Therefore, $\int_{-1}^{1} xP_n(x)\,P_{n-1}(x)\,dx$

$$= \frac{1}{2n-1}\int_{-1}^{1} [nP_n^{\ 2} + (n-1)\,P_n\,P_{n-2}]\,dx$$

$$= \frac{n}{2n-1}\left(\frac{2}{2n+1}\right) \qquad \Big|\ \text{Using orthogonal property}$$

$$= \frac{2n}{4n^2-1}.$$

Example 31. *Prove that:* $\int_{-1}^{1} [P_n'(x)]^2 \, dx = n(n+1)$.

Sol. $\int_{-1}^{1} [P_n'(x)]^2 \, dx$

$$= \int_{-1}^{1} [(2n-1)P_{n-1} + (2n-5)P_{n-3} + (2n-9)P_{n-5} + \ldots]^2 \, dx$$

| By Cristoffel's expansion formula

$$= \int_{-1}^{1} (2n-1)^2 P^2_{n-1} \, dx + \int_{-1}^{1} (2n-5)^2 P^2_{n-3} \, dx + \int_{-1}^{1} (2n-9)^2 P^2_{n-5} \, dx + \ldots$$

$$+ 2\int_{-1}^{1} (2n-1)(2n-5) P_{n-1} P_{n-3} dx + 2\int_{-1}^{1} (2n-1)(2n-9) P_{n-1} P_{n-5} \, dx + \ldots$$

$$= (2n-1)^2 \cdot \frac{2}{2(n-1)+1} + (2n-5)^2 \cdot \frac{2}{2(n-3)+1} + (2n-9)^2 \cdot \frac{2}{2(n-5)+1} + \ldots\ldots + 0 + 0 + 0 + \ldots$$

| By orthogonal properties

$$= (2n-1)^2 \cdot \frac{2}{2n-1} + (2n-5)^2 \cdot \frac{2}{2n-5} + (2n-9)^2 \cdot \frac{2}{2n-9} + \ldots$$

$$= 2\,[(2n-1) + (2n-5) + (2n-9) + \ldots + 1]$$

Here $\quad l = a + (N-1)d$

$$1 = (2n-1) + (N-1)(-4) \quad \Rightarrow \quad N = \frac{n+1}{2}$$

$\therefore$ No. of terms in above series $= \dfrac{n+1}{2}$

$$\therefore \quad S_{\frac{n+1}{2}} = 2 \cdot \frac{1}{2}\left(\frac{n+1}{2}\right)[(2n-1)+1] = n(n+1) \qquad \Big| \because \; S_n = \frac{n}{2}(a+l)$$

Hence, $\int_{-1}^{1} [P_n'(x)]^2 \, dx = n(n+1)$.

Example 32. *Prove that* $(1 - 2xz + z^2)^{-1/2}$ *is a solution of the equation*

$$z \frac{\partial^2}{\partial z^2}(z\nu) + \frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial \nu}{\partial x}\right] = 0.$$

Sol. We have, $\quad \nu = (1 - 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x) \quad$ or $\quad z\nu = \sum_{n=0}^{\infty} z^{n+1} P_n(x)$

$$\therefore \quad z\frac{\partial^2}{\partial z^2}(z\nu) = \sum_{n=0}^{\infty} (n+1)n z^n P_n(x) \quad \ldots(1)$$

Also, $$\frac{\partial \nu}{\partial x} = \sum_{n=0}^{\infty} z^n P_n'(x)$$

$$\therefore \quad \frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial \nu}{\partial x}\right\} = \frac{\partial}{\partial x}\left[(1-x^2)\sum_{n=0}^{\infty} z^n P_n'(x)\right]$$

$$= (1-x^2)\sum_{n=0}^{\infty} z^n P_n''(x) - 2x\sum_{n=0}^{\infty} z^n P_n'(x) \quad \ldots(2)$$

Substituting in LHS of the given equation, we have

$$\sum_{n=0}^{\infty} [(n+1)n\, z^n\, P_n(x) + (1-x^2)\, z^n\, P_n''(x) - 2xz^n\, P_n'(x)]$$

$$= \sum_{n=0}^{\infty} z^n\, [(1-x^2)\, P_n''(x) - 2x\, P_n'(x) + n(n+1)\, P_n(x)]$$

$= 0.$ | $\because$ $P_n(x)$ is a solution of Legendre's equation

EXERCISE

1. Show that:

(i) $x^2 = \frac{1}{3} P_0(x) + \frac{2}{3} P_2(x)$ (ii) $x^3 = \frac{2}{5} P_3(x) + \frac{3}{5} P_1(x)$

(iii) $x^5 = \frac{8}{63}\left[P_5(x) + \frac{7}{2} P_3(x) + \frac{27}{8} P_1(x)\right]$

2. Express the following in terms of Legendre's polynomials:

(i) $1 + x - x^2$ (ii) $x^4 + 3x^3 - x^2 + 5x - 2$

(iii) $5x^3 + x$ (iv) $x^3 - 5x^2 + 6x + 1$

(v) $4x^3 - 2x^2 - 3x + 8$ (vi) $x^4 + 2x^3 + 2x^2 - x - 3$

(vii) $2x^3 + 2x^2 - x - 3$

3. Expand $x^4 - 3x^2 + x$ in a series of the form $\Sigma C_r P_r(x)$.

4. Expand $f(x)$ in a series of Legendre polynomials if $f(x) = \left.\begin{matrix} 0, & -1 \le x < 0 \\ 2x+1, & 0 < x \le 1 \end{matrix}\right\}$.

5. Obtain the Fourier-Legendre expansion of $f(x)$ defined as

$$f(x) = \left.\begin{matrix} 0, & -1 < x < 0 \\ 1, & 0 < x < 1 \end{matrix}\right\}.$$

6. Express the function $f(x) = \left.\begin{matrix} 0, & -1 < x \le 0 \\ x^2, & 0 < x < 1 \end{matrix}\right\}$ in Fourier-Legendre expansion.

7. Expand $f(x) = \cos \frac{\pi x}{2}$ in Fourier-Legendre series.

8. Prove that:

(i) $\int_0^1 P_{2n}(x)\, P_{2n+1}(x)\, dx = \int_0^1 P_{2n}(x)\, P_{2n-1}(x)\, dx$ (ii) $\int_{-2}^1 x^6\, P_4(x)\, dx = \frac{16}{231}$

(iii) $\int_0^1 P_{2n+1}(x)\, dx = (-1)^n$.

9. Prove that:

(i) $\int_0^1 P_n(x) dx = \frac{1}{n+1} P_{n-1}(0)$ (ii) $\int_{-1}^1 P_3^2(x) = \frac{2}{7}$

(iii) $\int_{-1}^1 P'_{n-1}(x)\, P'_{n+1}(x)\, dx = n(n-1)$ (iv) $\int_0^1 P_{2n}(x)\, dx = 0$

Answers

2. (i) $\frac{2}{3} P_0(x) + P_1(x) - \frac{2}{3} P_2(x)$

(ii) $\frac{8}{35} P_4(x) + \frac{6}{5} P_3(x) - \frac{2}{21} P_2(x) + \frac{34}{5} P_1(x) - \frac{224}{105} P_0(x)$ (iii) $2P_3(x) + 4P_1(x)$

(iv) $\frac{2}{5} P_3(x) - \frac{10}{3} P_2(x) + \frac{33}{5} P_1(x) - \frac{2}{3} P_0(x)$

(*v*) $\frac{22}{3}\,P_0(x) - \frac{3}{5}\,P_1(x) - \frac{4}{3}\,P_2(x) + \frac{8}{5}\,P_3(x)$

(*vi*) $\frac{8}{35}\,P_4(x) + \frac{4}{5}\,P_3(x) + \frac{40}{21}\,P_2(x) + \frac{1}{5}\,P_1(x) - \frac{224}{105}\,P_0(x)$

(*vii*) $\frac{4}{5}P_3(x) + \frac{4}{3}P_2(x) + \frac{1}{5}P_1(x) - \frac{7}{3}P_0(x)$

3. $-\frac{4}{5}P_0(x) + P_1(x) - \frac{10}{7}P_2(x) + \frac{8}{35}P_4(x)$

4. $P_0(x) + \frac{7}{4}\,P_1(x) + \frac{5}{8}P_2(x) - \frac{7}{16}\,P_3(x) + \cdots$

5. $\frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) - \frac{7}{16}P_3(x) + \cdots$

6. $\frac{1}{6}\,P_0(x) + \frac{3}{8}\,P_1(x) + \frac{1}{3}\,P_2(x) + \frac{7}{48}\,P_3(x) + \cdots$

7. $\cos\frac{\pi x}{2} = 0.6366\,P_0 - 0.6871\,P_2 + .0518\,P_4 - .0013\,P_6 + \cdots$

NOTES

7. BESSEL'S DIFFERENTIAL EQUATION

STRUCTURE

Introduction
Solution of Bessel's equation
Series Representation of Bessel functions
Recurrence Relations for $J_n(x)$
Generating function for $Jn(x)$
Integral Form of Bessel Function
Equations Reducible to Bessel's Equation
Modified Bessel's Equation
BER and BEI Functions
Orthogonality of Bessel Functions
Fourier-Bessel Expansion of $F(x)$

INTRODUCTION

The differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

is called Bessel's differential equation of order n, where n is a positive constant.

This equation can also be put in the form

$$x\frac{d}{dx}\left(x\frac{dy}{dx}\right) + (x^2 - n^2)\, y = 0$$

The particular solutions of this equation are called Bessel's functions of order n.

SOLUTION OF BESSEL'S EQUATION

Bessel's equation is

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \qquad \ldots(1)$$

Comparing equation (1) with the form $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0$, we get

$$P(x) = \frac{1}{x} \text{ and } Q(x) = 1 - \frac{n^2}{x^2}$$

At $x = 0$, both $P(x)$ and $Q(x)$ are not analytic $\quad\therefore\quad x = 0$ is a singular point.

Also, $xP(x) = 1$ and $x^2\,Q(x) = x^2 - n^2$

Since both $xP(x)$ and $x^2\,Q(x)$ are analytic at $x = 0$

$\therefore\quad x = 0$ is a regular singular point.

Assume $$y = \sum_{k=0}^{\infty} a_k\, x^{m+k}$$

Then, $$\frac{dy}{dx} = \sum_{k=0}^{\infty} (m+k) a_k\, x^{m+k-1}$$

and $$\frac{d^2y}{dx^2} = \sum_{k=0}^{\infty} (m+k)(m+k-1)\, a_k\, x^{m+k-2}$$

Substituting for y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1), we get

$$x^2 \sum_{k=0}^{\infty} (m+k)(m+k-1) a_k\, x^{m+k-2} + x \sum_{k=0}^{\infty} (m+k)\, a_k\, x^{m+k-1}$$

$$+ (x^2 - n^2) \sum_{k=0}^{\infty} a_k\, x^{m+k} = 0$$

or $$\sum_{k=0}^{\infty} [(m+k)^2 - (m+k) + (m+k) - n^2]\, a_k\, x^{m+k} + \sum_{k=0}^{\infty} a_k\, x^{m+k+2} = 0$$

or $$\sum_{k=0}^{\infty} [(m+k)^2 - n^2]\, a_k\, x^{m+k} + \sum_{k=0}^{\infty} a_k\, x^{m+k+2} = 0$$

The lowest power of x is x^m corresponding to $k = 0$. Equating to zero the coefficient of x^m, we get the indicial equation

$$m^2 - n^2 = 0, \text{ since } a_0 \neq 0 \quad \text{whence } m = \pm n$$

Equating to zero the coefficient of next term *i.e.,* x^{m+1}, we get

$$[(m+1)^2 - n^2]a_1 = 0$$

$\Rightarrow \quad a_1 = 0$, since $(m+1)^2 - n^2 \neq 0$ for $m = \pm n$

Equating to zero the coefficient of x^{m+k+2}, we get the recurrence relation

$$[(m+k+2)^2 - n^2]\, a_{k+2} + a_k = 0$$

or $$a_{k+2} = -\frac{a_k}{(m-n+k+2)(m+n+k+2)}$$

Putting $k = 1, 3, 5, ...$, we get $a_3 = a_5 = a_7 = ... = 0$

Putting $k = 0, 2, 4, ...$, we get

$$a_2 = -\frac{a_0}{(m-n+2)(m+n+2)}$$

$$a_4 = -\frac{a_2}{(m-n+4)(m+n+4)} = \frac{a_0}{(m-n+4)(m+n+4)(m-n+2)(m+n+2)}$$

and so on.

NOTES

$$\therefore \quad \boxed{y = a_0 x^m \left[1 - \frac{x^2}{(m+2)^2 - n^2} + \frac{x^4}{[(m+2)^2 - n^2][(m+4)^2 - n^2]} - \cdots\right]} \quad \ldots(2)$$

Depending upon the values of n, we get different types of solutions.

Case I. *When $n \neq 0$ or $n \neq$ an integer.*

In this case, we get two independent solutions for $m = n$ and $m = -n$.

For $m = n$, we get

$$y_1 = a_0 x^n \left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2 \cdot 4(2n+2)(2n+4)} - \ldots\ldots\right]$$

$$= a_0 x^n \left[1 + (-1)^1 \frac{x^2}{2^2 (1)!(n+1)} + (-1)^2 \frac{x^4}{2^4 (2)!(n+1)(n+2)} + \cdots\right]$$

$$= a_0 x^n \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k} (k)!(n+1)(n+2)\ldots\ldots(n+k)} x^{2k}$$

$$\Rightarrow \quad y_1 = a_0 x^n \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(n+1)}{2^{2k} . k! \Gamma(n+k+1)} x^{2k} \quad \ldots(3)$$

Replacing n by $-n$, the second independent solution corresponding to $m = -n$ is

$$y_2 = a_0 x^n \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(-n+1)}{2^{2k} (k)! \Gamma(-n+k+1)} x^{2k} \quad \ldots(4)$$

$\therefore$ The complete solution of equation (1) is $y = c_1 y_1 + c_2 y_2$

Since a_0 is arbitrary, we can choose it in any manner.

Choose $a_0 = \dfrac{1}{2^n \Gamma(n+1)}$, then (3) takes the form

$$y_1 = \frac{x^n}{2^n \Gamma(n+1)} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(n+1)}{2^{2k} . k! \Gamma(n+k+1)} x^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k)! \Gamma(n+k+1)} \left(\frac{x}{2}\right)^{n+2k}$$

This is called **Bessel function of the first kind of order *n*** and is denoted by **$J_n(x)$.** Thus,

$$\boxed{J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k)! \Gamma(n+k+1)} \left(\frac{x}{2}\right)^{n+2k}}$$

The solution corresponding to $m = -n$ is

$$\boxed{J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k)! \Gamma(-n+k+1)} \left(\frac{x}{2}\right)^{-n+2k}}$$

which is called **Bessel function of the first kind of order – *n*.**

When n is not an integer, $J_{-n}(x)$ is distinct from $J_n(x)$. Hence the complete solution of the Bessel's equation may be expressed as

$$\boxed{y = AJ_n(x) + BJ_{-n}(x)}$$, where A and B are arbitrary constants.

Case II. *When n = 0,* the Bessel's equation (1) takes the form

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+xy=0.$$

This is called **Bessel's equation of order zero.**

The two roots of the indicial equation are equal, each = 0.

From equation (2), putting $n = 0$, we have (assuming $a_0 = 1$)

$$y=x^m\left[1-\frac{x^2}{(m+2)^2}+\frac{x^4}{(m+2)^2(m+4)^2}-\frac{x^6}{(m+2)^2(m+4)^2(m+6)^2}+\dots\dots\right]$$

which is a solution if $m = 0$.

The first solution is given by

$$\boxed{J_0(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(k\,!)^2}\left(\frac{x}{2}\right)^{2k}},\qquad \text{since } \Gamma(k+1)=k\,!$$

which is **Bessel function of the first kind of order zero.**

Now,
$$\frac{\partial y}{\partial m}=x^m\log x\left[1-\frac{x^2}{(m+2)^2}+\frac{x^4}{(m+2)^2(m+4)^2}-\dots\dots\right]$$
$$+x^m\left[\frac{x^2}{(m+2)^2}\cdot\frac{2}{m+2}-\frac{x^4}{(m+2)^2(m+4)^2}\left\{\frac{2}{m+2}+\frac{2}{m+4}\right\}+\dots\dots\right]$$

The second independent solution is given by $\left(\frac{\partial y}{\partial m}\right)_{m=0}$

$$=J_0(x)\log x+\left[\frac{1}{2^2}x^2-\frac{1}{2^2.4^2}\left(1+\frac{1}{2}\right)x^4+\frac{1}{2^2.4^2.6^2}\left(1+\frac{1}{2}+\frac{1}{3}\right)x^6-\dots\dots\right]$$

$$=J_0(x)\log x+\left[\left(\frac{x}{2}\right)^2-\frac{1}{(2\,!)^2}\left(1+\frac{1}{2}\right)\left(\frac{x}{2}\right)^4+\frac{1}{(3\,!)^2}\left(1+\frac{1}{2}+\frac{1}{3}\right)\left(\frac{x}{2}\right)^6-\dots\dots\right]$$

$$=J_0(x)\log x+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(k\,!)^2}\left(1+\frac{1}{2}+\frac{1}{3}+\dots\dots+\frac{1}{k}\right)\left(\frac{x}{2}\right)^{2k}$$

It is denoted by $Y_0(x)$ and is called **Bessel function of the second kind of order zero or Neumann function.**

Thus the complete solution of the Bessel's equation of order zero is

$$\boxed{y=AJ_0(x)+BY_0(x)}$$

Case III. *When n is an integer,* the two functions $J_n(x)$ and $J_{-n}(x)$ are not independent but are connected by the relation

$$\mathbf{J_{-n}(x)=(-1)^n\,J_n(x).}$$

Now, when n is an integer, y_2 fails to give a solution for positive values of n and y_1 fails to give a solution for negative values of n. Let us find an independent solution of Bessel's equation (1), when n is an integer.

Let $y = u(x)\,J_n(x)$ be a solution of (1) when n is integral.

Then
$$\frac{dy}{dx}=u'J_n+uJ_n' \quad\text{and}\quad \frac{d^2y}{dx^2}=u''J_n+2u'J_n'+uJ_n''$$

NOTES

Substituting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1), we get

$$x^2(u''J_n + 2u'J_n' + uJ_n'') + x(u'J_n + uJ_n') + (x^2 - n^2)uJ_n = 0$$

or $$u[x^2J_n'' + xJ_n' + (x^2 - n^2)J_n] + x^2u''J_n + 2x^2u'J_n' + xu'J_n = 0$$

or $$x^2u''J_n + 2x^2u'J_n' + xu'J_n = 0 \text{ since } J_n \text{ is a solution of (1).}$$

Dividing throughout by $x^2u'J_n$, we get

$$\frac{u''}{u'} + 2\frac{J_n'}{J_n} + \frac{1}{x} = 0$$

Integrating w.r.t. x, we get

$$\log (u'J_n^2x) = \log B$$

or $$u' J_n^2x = B$$

or $$u' = \frac{B}{xJ_n^2} \quad \text{or} \quad u = B\int \frac{dx}{xJ_n^2} + A$$

Substituting the value of u in the assumed solution $y = u(x)\, J_n(x)$, we have

$$y = \left[B\int \frac{dx}{xJ_n^2(x)} + A\right] J_n(x)$$

or $$y = AJ_n(x) + BY_n(x), \text{ where } \quad Y_n(x) = J_n(x)\int \frac{dx}{xJ_n^2(x)}$$

The function $Y_n(x)$ is called the **Bessel function of the second kind of order n or Neumann function.**

SERIES REPRESENTATION OF BESSEL FUNCTIONS

Since $$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(n+k+1)}\left(\frac{x}{2}\right)^{n+2k}$$

$\therefore$ $$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+1)}\left(\frac{x}{2}\right)^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2}\left(\frac{x}{2}\right)^{2k}, \because \Gamma(k+1) = k!$$

$$= 1 - \left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \cdots\cdots$$

$$= 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \cdots\cdots$$

$$J_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+2)}\left(\frac{x}{2}\right)^{1+2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+1)!}\left(\frac{x}{2}\right)^{1+2k}$$

$$= \frac{x}{2} - \frac{1}{2!}\left(\frac{x}{2}\right)^3 + \frac{1}{2!3!}\left(\frac{x}{2}\right)^5 - \cdots\cdots = \frac{x}{2} - \frac{x^3}{2^2 \cdot 4} + \frac{x^5}{2^2 \cdot 4^2 \cdot 6} - \cdots\cdots$$

In particular, $J_0(0) = 1$ and $J_1(0) = 0$.

The values of $J_0(x)$ and $J_1(x)$ are given in 'Jahnke Emde's tables' to four decimal places at intervals of 0.1.

RECURRENCE RELATIONS FOR $J_n(x)$

$$x\,J_n' = nJ_n - x\,J_{n+1}$$

We know that,

$$J_n = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma n + r + 1}\left(\frac{x}{2}\right)^{n+2r}$$

Differentiating w.r.t. x, we get

$$J_n' = \sum_{r=0}^{\infty} \frac{(-1)^r\,(n+2r)}{r!\,\Gamma n + r + 1}\cdot\frac{1}{2}\left(\frac{x}{2}\right)^{n+2r-1}$$

Multiplying both sides by x and breaking it into two terms

$$xJ_n' = n\sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma n + r + 1}\left(\frac{x}{2}\right)^{n+2r} + x\sum_{r=1}^{\infty} \frac{(-1)^r}{(r-1)!\,\Gamma n + r + 1}\left(\frac{x}{2}\right)^{n+2r-1}$$

$$= nJ_n + x\sum_{s=0}^{\infty} \frac{(-1)^{s+1}}{s!\,\Gamma n + s + 2}\left(\frac{x}{2}\right)^{n+2s+1} \qquad | \text{ Here } r = s + 1$$

$$= nJ_n - x\sum_{s=0}^{\infty} \frac{(-1)^{s}}{s!\,\Gamma n + s + 2}\left(\frac{x}{2}\right)^{(n+1)+2s}$$

$$\Rightarrow \qquad x\,J_n' = nJ_n - xJ_{n+1} \qquad \text{...(1)}$$

$$xJ_n' = -\,nJ_n + xJ_{n-1}$$

We know that

$$xJ_n' = \sum_{r=0}^{\infty} \frac{(-1)^r\,(n+2r)}{r!\,\Gamma n + r + 1}\left(\frac{x}{2}\right)^{n+2r} = \sum_{r=0}^{\infty} \frac{(-1)^r\,(2n+2r-n)}{r!\,\Gamma n + r + 1}\left(\frac{x}{2}\right)^{n+2r}$$

$$= -\,n\sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma n + r + 1}\left(\frac{x}{2}\right)^{n+2r} + x\sum_{r=0}^{\infty} \frac{(-1)^r\,(n+r)}{r!\,\Gamma n + r + 1}\left(\frac{x}{2}\right)^{n+2r-1}$$

$$= -\,nJ_n + x\sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma n + r}\left(\frac{x}{2}\right)^{n+2r-1}$$

$$= -\,nJ_n + x\sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n-1) + r + 1}\left(\frac{x}{2}\right)^{n-1+2r}$$

$$xJ_n' = -\,nJ_n + xJ_{n-1} \qquad \text{...(2)}$$

$$2J_n' = J_{n-1} - J_{n+1}$$

Adding equations (1) and (2), we get

$$2xJ_n' = x\,(J_{n-1} - J_{n+1})$$

$$\Rightarrow \qquad 2J_n' = J_{n-1} - J_{n+1} \qquad \text{...(3)}$$

NOTES

$$\boxed{2nJ_n = x(J_{n-1} + J_{n+1})}$$

Subtracting (2) from (1), we get

$$0 = 2n\, J_n - x\, J_{n-1} - xJ_{n+1}$$

$$\Rightarrow \qquad 2nJ_n = x\,(J_{n-1} + J_{n+1}) \qquad \text{...(4)}$$

$$\boxed{\frac{d}{dx}(x^{-n}\, J_n) = -\, x^{-n}\, J_{n+1}}$$

Multiplying eqn. (1) by x^{-n-1}, we get

$$x^{-n}\, J_n' = nx^{-n-1}\, J_n - x^{-n}\, J_{n+1}$$

$$\Rightarrow \qquad x^{-n}\, J_n' - x^{-n-1}\, .\, nJ_n = -\, x^{-n}\, J_{n+1}$$

$$\frac{d}{dx}(x^{-n}\, J_n) = -\, x^{-n}\, J_{n+1} \qquad \text{...(5)}$$

$$\boxed{\frac{d}{dx}(x^{n}\, J_n) = x^{n}\, J_{n-1}}$$

Multiplying eqn. (2) by x^{n-1}, we get

$$x^n\, J_n' = -\, n\, x^{n-1}\, J_n + x^n\, J_{n-1}$$

$$\Rightarrow \qquad x^n\, J_n' + nx^{n-1}\, J_n = x^n\, J_{n-1}$$

$$\Rightarrow \qquad \frac{d}{dx}(x^n\, J_n) = x^n\, J_{n-1} \qquad \text{...(6)}$$

SOLVED EXAMPLES

Example 1. *Prove that:* $J_{-n}(x) = (-1)^n\, J_n(x)$.

Sol. Since $\Gamma - p$ is infinity ($p > 0$), we get terms in $J_{-n}(x)$ equal to zero till $r + 1 - n \geq 1$ so that the series begins when $r \geq n$

Hence we can write,

$$J_{-n}(x) = \sum_{r=n}^{\infty} \frac{(-1)^r}{r!\,\Gamma - n + r + 1}\left(\frac{x}{2}\right)^{-n+2r}$$

Putting $r = n + s$, we get

$$J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^{n+s}}{(n+s)!\,\Gamma s + 1}\left(\frac{x}{2}\right)^{n+2s}$$

$$= (-1)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{(n+r)!\,\Gamma r + 1}\left(\frac{x}{2}\right)^{n+2r} = (-1)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma n + r + 1\;\, r!}\left(\frac{x}{2}\right)^{n+2r}$$

$$\Rightarrow \qquad J_{-n}(x) = (-1)^n\, J_n(x).$$

Example 2. *Prove that:* $J_0'(x) = -\, J_1(x)$.

Sol. We know that $\quad \dfrac{d}{dx}[x^{-n}\, J_n(x)] = -\, x^{-n}\, J_{n+1}(x)$

Putting $n = 0$, we get $\quad \dfrac{d}{dx}[J_0(x)] = -\, J_1(x) \quad \Rightarrow \quad J_0'(x) = -\, J_1(x).$

Example 3. *Prove that:*

(*i*) $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$ (*ii*) $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$.

Sol. We know that,

$$J_n(x) = \frac{x^n}{2^n \Gamma n+1}\left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2\cdot 4\cdot(2n+2)(2n+4)} - \ldots\ldots\right] \quad \ldots(1)$$

(*i*) Putting $n = \frac{1}{2}$ in (1),

$$J_{1/2}(x) = \frac{\sqrt{x}}{\sqrt{2}\,\Gamma 3/2}\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots\right]$$

$$= \frac{\sqrt{x}}{\sqrt{2}\,.\,\frac{1}{2}\sqrt{\pi}}\cdot\frac{1}{x}\left[x - \frac{x^3}{3!} + \frac{x^3}{5!} - \ldots\right] = \sqrt{\frac{2}{\pi x}} \sin x$$

(*ii*) Putting $n = -\frac{1}{2}$ in (1),

$$J_{-1/2}(x) = \frac{x^{-1/2}}{2^{-1/2}\,\Gamma 1/2}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) = \sqrt{\frac{2}{\pi x}} \cos x\,.$$

Example 4. *Prove that:*

$$\frac{d}{dx}[J_n^2(x) + J^2_{n+1}(x)] = 2\left[\frac{n}{x}J_n^2(x) - \frac{n+1}{x}J^2_{n+1}(x)\right].$$

Sol. $\text{LHS} = 2J_n J_n' + 2J_{n+1} J'_{n+1}$ ...(1)

But $xJ_n' = nJ_n - xJ_{n+1}$ | Recurrence relation (1)

$\therefore$ $J_n' = \frac{n}{x} J_n - J_{n+1}$ ...(2)

and also, $xJ_n' = -nJ_n + xJ_{n-1}$ | Recurrence relation (2)

$\therefore$ $J_n' = -\frac{n}{x} J_n + J_{n-1}$

or $J'_{n+1} = -\left(\frac{n+1}{x}\right) J_{n+1} + J_n$ ...(3)

Substituting these values of J_n' and J'_{n+1} from (2) and (3) in eqn. (1), we get

$$\text{LHS} = 2J_n\left(\frac{n}{x}J_n - J_{n+1}\right) + 2J_{n+1}\left(-\frac{n+1}{x}J_{n+1} + J_n\right)$$

$$= 2\,\frac{n}{x}J_n^2 - 2\left(\frac{n+1}{x}\right)J^2_{n+1} = \text{RHS}$$

Hence the result.

Example 5. *Prove that:*

(*i*) $J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$ (*ii*) $J_{-3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{-\cos x}{x} - \sin x\right)$.

NOTES

Sol. By Recurrence relation (4), we have

$$2n\,J_n(x) = x\,[J_{n-1}(x) + J_{n+1}(x)] \quad ...(1)$$

$$\Rightarrow \quad J_{n+1}(x) = \frac{2n}{x}\,J_n(x) - J_{n-1}(x) \quad ...(2)$$

(*i*) Putting $n = 1/2$ in (2), we get

$$J_{3/2}(x) = \frac{1}{x}\,J_{1/2}(x) - J_{-1/2}(x)$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{\sin x}{x} - \cos x\right] \quad \text{| Using results of Ex. 3}$$

(*ii*) From equation (1),

$$J_{n-1}(x) = \frac{2n}{x}\,J_n(x) - J_{n+1}(x) \quad ...(3)$$

Putting $n = -1/2$ in (3), we get

$$J_{-3/2}(x) = -\frac{1}{x}\,J_{-1/2}(x) - J_{1/2}(x)$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{-\cos x}{x} - \sin x\right] \quad \text{| Using results of Ex. 3}$$

Example 6. *Prove that:* $J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\left(\frac{3 - x^2}{x^2}\right)\sin x - \frac{3\cos x}{x}\right]$

Sol. From Recurrence relation (4),

$$2nJ_n(x) = x[J_{n-1}(x) + J_{n+1}(x)]$$

$$\Rightarrow \quad J_{n+1}(x) = \frac{2n}{x}\,J_n(x) - J_{n-1}(x) \quad ...(1)$$

Putting $n = 1/2, 3/2$ in (1), we get

$$J_{3/2}(x) = \frac{1}{x}\,J_{1/2}(x) - J_{-1/2}(x) \quad ...(2)$$

$$J_{5/2}(x) = \frac{3}{x}\,J_{3/2}(x) - J_{1/2}(x) \quad ...(3)$$

From (2) and (3),

$$J_{5/2}(x) = \frac{3}{x}\left[\frac{1}{x}\,J_{1/2}(x) - J_{-1/2}(x)\right] - J_{1/2}(x)$$

$$= \left(\frac{3}{x^2} - 1\right)J_{1/2}(x) - \frac{3}{x}\,J_{-1/2}(x) = \left(\frac{3 - x^2}{x^2}\right)\sqrt{\frac{2}{\pi x}}\sin x - \frac{3}{x}\cdot\sqrt{\frac{2}{\pi x}}\cos x$$

$$= \sqrt{\frac{2}{\pi x}}\left[\left(\frac{3 - x^2}{x^2}\right)\sin x - \frac{3}{x}\cos x\right].$$

Example 7. *Prove that:* $J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right)J_1(x) + \left(1 - \frac{24}{x^2}\right)J_0(x)$.

Hence or otherwise find $J_6(x)$ *in terms of* $J_0(x)$ *and* $J_1(x)$.

Sol. From Recurrence relation (4),

$$2nJ_n(x) = x\,[J_{n-1}(x) + J_{n+1}(x)]$$

$$\Rightarrow \quad J_{n+1}(x) = \frac{2n}{x}\,J_n(x) - J_{n-1}(x) \quad ...(1)$$

Putting $n = 1, 2, 3$ in eqn. (1), we get

$$J_2(x) = \frac{1}{x}[2J_1(x) - xJ_0(x)] \quad ...(2)$$

$$J_3(x) = \frac{1}{x}[4J_2(x) - xJ_1(x)] \quad ...(3)$$

$$J_4(x) = \frac{1}{x}[6J_3(x) - xJ_2(x)] \quad ...(4)$$

From (2) and (3),

$$J_3(x) = \frac{8}{x^2}J_1(x) - \frac{4}{x}J_0(x) - J_1(x)$$

$$= \left(\frac{8 - x^2}{x^2}\right)J_1(x) - \frac{4}{x}J_0(x) \quad ...(5)$$

Again from (4) and (5),

$$J_4(x) = \left(\frac{48 - 6x^2}{x^3}\right)J_1(x) - \frac{24}{x^2}J_0(x) - \frac{2}{x}J_1(x) + J_0(x)$$

$$= \left(\frac{48}{x^3} - \frac{8}{x}\right)J_1(x) + \left(1 - \frac{24}{x^2}\right)J_0(x). \quad ...(6)$$

Again, putting $n = 4, 5$ in eqn. (1), we get

$$J_5(x) = \frac{8}{x}J_4(x) - J_3(x) \quad ...(7)$$

$$J_6(x) = \frac{10}{x}J_5(x) - J_4(x) \quad ...(8)$$

From (7) and (8),

$$J_6(x) = \frac{10}{x}\left[\frac{8}{x}J_4(x) - J_3(x)\right] - J_4(x)$$

$$= \left(\frac{80}{x^2} - 1\right)J_4(x) - \frac{10}{x}J_3(x)$$

$$= \left(\frac{80}{x^2} - 1\right)\left[\left(\frac{48}{x^3} - \frac{8}{x}\right)J_1(x) + \left(1 - \frac{24}{x^2}\right)J_0(x)\right]$$

$$- \frac{10}{x}\left[\left(\frac{8}{x^2} - 1\right)J_1(x) - \frac{4}{x}J_0(x)\right] \quad | \text{ Using eqn. (5) and (6)}$$

$$\Rightarrow \quad J_6(x) = \left(\frac{3840}{x^5} - \frac{768}{x^3} + \frac{18}{x}\right)J_1(x) + \left(\frac{144}{x^2} - \frac{1920}{x^4} - 1\right)J_0(x).$$

Example 8. *Prove that:* $\int J_3(x)\,dx + J_2(x) + \frac{2}{x}J_1(x) = 0.$

Sol. We know that,

$$\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$$

$$\therefore \int x^{-n}J_{n+1}(x)\,dx = -x^{-n}J_n(x) \quad ...(1)$$

NOTES

Now, $\int J_3(x)\,dx = \int x^2 [x^{-2} J_3(x)]\,dx$

$$= x^2 . [-x^{-2} J_2(x)] - \int 2x . [-x^{-2} J_2(x)]\,dx$$

$$= -J_2(x) + 2\int x^{-1} J_2(x)dx$$

$$= -J_2(x) + 2[-x^{-1} J_1(x)] = -J_2(x) - \frac{2}{x} J_1(x)$$

or $\int J_3(x)\,dx + J_2(x) + \frac{2}{x} J_1(x) = 0.$

Example 9. *Prove that:* $\int xJ_0^2(x)\,dx = \frac{1}{2} x^2 [J_0^2(x) + J_1^2(x)] + c.$

Sol. $\int xJ_0^2(x)\,dx = J_0^2(x) . \frac{x^2}{2} - \int 2J_0(x) J_0'(x) . \frac{x^2}{2}\,dx + c$

$$= \frac{x^2}{2} J_0^2(x) - \int x^2 J_0(x) \{-J_1(x)\}\,dx + c \quad [\because \quad J_0'(x) = -J_1(x)]$$

$$= \frac{x^2}{2} J_0^2(x) + \int xJ_1(x) . xJ_0(x)\,dx + c$$

$$= \frac{x^2}{2} J_0^2(x) + \int xJ_1(x) . \frac{d}{dx}[xJ_1(x)]\,dx + c$$

| Using Recurrence relation

$$= \frac{x^2}{2} J_0^2(x) + \frac{[xJ_1(x)]^2}{2} + c = \frac{x^2}{2}[J_0^2(x) + J_1^2(x)] + c.$$

Example 10. *Prove that:* $4J_n''(x) = J_{n-2}(x) - 2J_n(x) + J_{n+2}(x).$

Sol. From Recurrence relation (3), we have

$$2J_n' = J_{n-1} - J_{n+1} \qquad \ldots(1)$$

Differentiating, $2J_n'' = J'_{n-1} - J'_{n+1}$

or $4J_n'' = 2J'_{n-1} - 2J'_{n+1} = (J_{n-2} - J_n) - (J_n - J_{n+2})$ | Using (1)

$\Rightarrow$ $4J_n'' = J_{n-2} - 2J_n + J_{n+2}$

Example 11. *Prove that:* $4J_0'''(x) + 3J_0'(x) + J_3(x) = 0.$

Sol. We know that

$$J_0' = -J_1$$

Differentiating gives, $J_0'' = -J_1' = -\frac{1}{2}(J_0 - J_2)$

[By Recurrence relation $2J_n' = J_{n-1} - J_{n+1}$]

Differentiating again, $J_0''' = -\frac{1}{2}(J_0' - J_2') = -\frac{1}{2}J_0' + \frac{1}{2} . \frac{1}{2}[J_1 - J_3]$

$$= -\frac{1}{2}J_0' + \frac{1}{4}J_1 - \frac{1}{4}J_3 = -\frac{1}{2}J_0' - \frac{1}{4}J_0' - \frac{1}{4}J_3 \qquad | \because \quad J_1 = -J_0'$$

$$= -\frac{3}{4}J_0' - \frac{1}{4}J_3$$

$\Rightarrow$ $4J_0''' + 3J_0' + J_3 = 0.$

Example 12. *Prove that:* $\frac{d}{dx}[xJ_n(x)\, J_{n+1}(x)] = x\,[J_n^2(x) - J^2_{n+1}(x)]$.

Sol. $$\text{LHS} = \frac{d}{dx}[x^{-n} J_n(x) \,.\, x^{n+1} J_{n+1}(x)]$$
$$= x^{-n} J_n(x) \frac{d}{dx}[x^{n+1} J_{n+1}(x)] + x^{n+1} J_{n+1}(x) \,.\, \frac{d}{dx}[x^{-n} J_n(x)]$$
$$= x^{-n} J_n(x) \cdot x^{n+1} J_n(x) + x^{n+1} J_{n+1}(x)\, [-x^{-n} J_{n+1}(x)]$$
$$\Big|\because \frac{d}{dx}(x^n J_n) = x^n J_{n-1}$$
$$= xJ_n^2(x) - xJ^2_{n+1} = x\,[J_n^2(x) - J^2_{n+1}] = \text{RHS}.$$

Example 13. *Prove that:* $\lim\limits_{x\to 0} \frac{J_n(x)}{x^n} = \frac{1}{2^n\,\Gamma n+1}$; $(n > -1)$.

Sol. We know that:

$$J_n(x) = \frac{x^n}{2^n\,\Gamma n+1}\left[1 - \frac{x^2}{2\cdot(2n+2)} + \frac{x^4}{2\cdot 4\cdot(2n+2)(2n+4)} - \ldots\ldots\right]$$

$$\therefore \quad \lim_{x\to 0}\frac{J_n(x)}{x^n} = \lim_{x\to 0}\frac{1}{2^n\,\Gamma n+1}\left[1 - \frac{x^2}{2\cdot(2n+2)} + \frac{x^4}{2\cdot 4\cdot(2n+2)(2n+4)} - \ldots\ldots\right]$$
$$= \frac{1}{2^n\,\Gamma n+1}.$$

Example 14. *Prove that:* $J_2'(x) = \left(1 - \frac{4}{x^2}\right) J_1(x) + \frac{2}{x} J_0(x)$.

Sol. By Recurrence relation (2), we have

$$xJ_n' = -nJ_n + xJ_{n-1} \qquad \ldots(1)$$

Putting $n = 2$, $\quad xJ_2' = -2J_2 + xJ_1$

$$\Rightarrow \qquad J_2' = -\frac{2}{x}J_2 + J_1 \qquad \ldots(2)$$

By Recurrence relation (1), we have

$$xJ_n' = nJ_n - xJ_{n+1} \qquad \ldots(3)$$

From (1) and (3), we have

$$-nJ_n + xJ_{n-1} = nJ_n - xJ_{n+1}$$

Putting $n = 1$, $\quad -J_1 + xJ_0 = J_1 - xJ_2$

$$\Rightarrow \qquad J_2 = \frac{2}{x}J_1 - J_0 \qquad \ldots(4)$$

$\therefore$ From (2), $$J_2' = -\frac{2}{x}\left(\frac{2}{x}J_1 - J_0\right) + J_1 = \left(1 - \frac{4}{x^2}\right)J_1 + \frac{2}{x}J_0.$$

Example 15. *Prove that:*

$$J_{n+3} + J_{n+5} = \frac{2}{x}\,(n+4)\,J_{n+4}.$$

Sol. By Recurrence relation (4), we have

$$2nJ_n = x\,(J_{n-1} + J_{n+1})$$

NOTES

Replacing n by $n + 4$, we get

$$\frac{2}{x}(n+4)\, J_{n+4} = J_{n+3} + J_{n+5}.$$

Example 16. *Prove that* $J_n(x) = 0$ *has no repeated root except at* $x = 0$.

Sol. Suppose, if possible, α is a double root of $J_n(x) = 0$

Then, $\qquad J_n(\alpha) = 0 \quad \text{and} \quad J_n'(\alpha) = 0 \qquad \ldots(1)$

From Recurrence relations, we know that

$$J_{n+1}(x) = \frac{n}{x} J_n(x) - J_n'(x)$$

and

$$J_{n-1}(x) = \frac{n}{x} J_n(x) + J_n'(x)$$

Using (1), we get $\quad J_{n+1}(\alpha) = 0 \quad \text{and} \quad J_{n-1}(\alpha) = 0$

which is inadmissible as power series cannot have the same sum function.

Hence $J_n(x)$ has no repeated root except $\quad x = 0$.

Example 17. *Prove that:*

$$x^2 J_n''(x) = (n^2 - n - x^2) J_n(x) + x J_{n+1}(x) ; n = 0, 1, 2.$$

Sol. We have

$$xJ_n' = nJ_n - xJ_{n+1} \qquad \ldots(1) \quad |\text{By R.R. (1)}$$

Diff., $\qquad xJ_n'' + J_n' = nJ_n' - xJ'_{n+1} - J_{n+1}$

$\Rightarrow \qquad x^2 J_n'' = (n-1)\, x J_n' - x^2 J'_{n+1} - xJ_{n+1} \qquad \ldots(2)$

By Recurrence relation (2),

$$xJ_n' = -nJ_n + xJ_{n-1}$$

$\Rightarrow \qquad xJ'_{n+1} = -(n+1) J_{n+1} + xJ_n \qquad \ldots(3)$

$\therefore$ From (2), $\quad x^2J_n'' = (n-1)[nJ_n - xJ_{n+1}] - x[-(n+1) J_{n+1} + xJ_n] - xJ_{n+1}$

$\Rightarrow \qquad x^2 J_n'' = (n^2 - n - x^2) J_n + x J_{n+1}.$

Example 18. *Prove that:* $\frac{x}{2} J_n = (n+1) J_{n+1} - (n+3) J_{n+3} + (n+5) J_{n+5} - \ldots$

Sol. By Recurrence relation (4), we have

$$2nJ_n = x(J_{n-1} + J_{n+1})$$

$\Rightarrow \qquad 2(n+1) J_{n+1} = x(J_n + J_{n+2}) \qquad |\text{ Replacing } n \text{ by } (n+1)$

$\Rightarrow \qquad \frac{x}{2} J_n = (n+1) J_{n+1} - \frac{x}{2} J_{n+2} \qquad \ldots(1)$

$\Rightarrow \qquad \frac{x}{2} J_{n+2} = (n+3) J_{n+3} - \frac{x}{2} J_{n+4} \qquad \ldots(2)$

$\therefore$ From (1) and (2),

$$\frac{x}{2} J_n = (n+1) J_{n+1} - (n+3) J_{n+3} + \frac{x}{2} J_{n+4}$$

Continuing this way,

$$\frac{x}{2} J_n = (n+1) J_{n+1} - (n+3) J_{n+3} + (n+5) J_{n+5} - \ldots$$

Example 19. *If* $n > -1$, *show that:* $\int_0^x x^{-n} J_{n+1}(x)\, dx = \frac{1}{2^n \Gamma n+1} - x^{-n} J_n(x)$.

Sol. We know that,

$$\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x) \qquad | \text{ Recurrence relation}$$

Integrating it between 0 and x, we get

$$\int_0^x x^{-n} J_{n+1}(x)\, dx = -\left[x^{-n} J_n(x)\right]_0^x = -x^{-n} J_n(x) + \underset{x \to 0}{\text{Lt}} \left[\frac{J_n(x)}{x^n}\right]$$

$$= -x^{-n} J_n(x) + \frac{1}{2^n \Gamma n+1}.$$

Example 20. *Prove that:* $J_n' = \frac{2}{x}\left[\frac{n}{2} J_n - (n+2) J_{n+2} + (n+4) J_{n+1} - \ldots\ldots\right]$.

Sol. From Recurrence formula (2), we have

$$J_n' = -\frac{n}{x} J_n + J_{n-1}$$

$$= -\frac{n}{x} J_n + \frac{2}{x}[nJ_n - (n+2) J_{n+2} + \cdots] \qquad | \text{Using example 18}$$

$$= \frac{2}{x}\left[\frac{n}{2} J_n - (n+2) J_{n+2} + \cdots\right].$$

GENERATING FUNCTION FOR $J_n(x)$

The function $e^{\frac{x}{2}\left(z - \frac{1}{z}\right)}$ is called generating function.

Prove that

(*i*) $$\boxed{e^{\frac{x}{2}\left(z - \frac{1}{z}\right)} = \sum_{n=-\infty}^{\infty} z^n J_n(x)}$$

i.e., $J_n(x)$ is the coefficient of z^n in the expansion of $e^{\frac{x}{2}\left(z - \frac{1}{z}\right)}$.

(*ii*) $$\boxed{e^{\frac{x}{2}\left(z - \frac{1}{z}\right)} = \sum_{n=-\infty}^{\infty} (-1)^n J_n(x) z^{-n}}$$

i.e., $(-1)^n J_n(x)$ is the coefficient of z^{-n} in the expansion of $e^{\frac{x}{2}\left(z - \frac{1}{z}\right)}$.

Note. Above results are true if n is an integer.

Proof. We have,

$$e^{\frac{x}{2}\left(z - \frac{1}{z}\right)} = e^{\frac{xz}{2}} e^{-\frac{x}{2z}}$$

$$= \left[1 + \frac{xz}{2} + \left(\frac{x}{2}\right)^2 \frac{z^2}{2!} + \ldots\ldots\right]\left[1 - \frac{x}{2z} + \left(\frac{x}{2z}\right)^2 \frac{1}{2!} - \ldots\ldots\right]$$

NOTES

(i) Coeff. of z^n in this product

$$= \left(\frac{x}{2}\right)^n \frac{1}{n!} - \left(\frac{x}{2}\right)^{n+2} \cdot \frac{1}{(n+1)!} + \left(\frac{x}{2}\right)^{n+4} \cdot \frac{1}{(n+2)!} \cdot \frac{1}{2!} + \ldots\ldots$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(n+r)!} \left(\frac{x}{2}\right)^{n+2r} = J_n$$

(ii) Coeff. of z^{-n} in this product

$$= \left(-\frac{x}{2}\right)^n \frac{1}{n!} - \left(-\frac{x}{2}\right)^{n+2} \cdot \frac{1}{(n+1)!} + \left(-\frac{x}{2}\right)^{n+4} \cdot \frac{1}{(n+2)!} \cdot \frac{1}{2!} + \ldots\ldots$$

$$= (-1)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(n+r)!} \left(\frac{x}{2}\right)^{n+2r} = (-1)^n J_n.$$

INTEGRAL FORM OF BESSEL FUNCTION

We know that $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$

$$= J_0(x) + tJ_1(x) + t^2J_2(x) + t^3J_3(x) + \ldots\ldots + t^{-1}J_{-1}(x) + t^{-2}J_{-2}(x) + t^{-3}J_{-3}(x) + \ldots\ldots$$

$$= J_0(x) + tJ_1(x) + t^2J_2(x) + t^3J_3(x) + \ldots\ldots - t^{-1}J_1(x) + t^{-2}J_2(x) - t^{-3}J_3(x) + \ldots\ldots$$

$[\because \quad J_{-n}(x) = (-1)^n J_n(x)]$

$$= J_0(x) + \left(t - \frac{1}{t}\right)J_1(x) + \left(t^2 + \frac{1}{t^2}\right)J_2(x) + \left(t^3 - \frac{1}{t^3}\right)J_3(x) + \ldots\ldots \qquad \ldots(1)$$

Put $t = \cos\theta + i\sin\theta$

$\therefore \quad t^n = \cos n\theta + i\sin n\theta$ and $\frac{1}{t^n} = \cos n\theta - i\sin n\theta$

| By De-Moivre's theorem

so that $t^n + \frac{1}{t^n} = 2\cos n\theta$ and $t^n - \frac{1}{t^n} = 2i\sin n\theta$

Substituting these values in (1), we have

$$e^{ix\sin\theta} = J_0(x) + 2i\sin\theta J_1(x) + 2\cos 2\theta\, J_2(x) + 3i\sin 3\theta\, J_3(x) + \ldots\ldots \qquad \ldots(2)$$

Since $e^{ix\sin\theta} = \cos(x\sin\theta) + i\sin(x\sin\theta)$

$\therefore$ Equating the real and imaginary parts in (2), we get

$$\mathbf{\cos(x\sin\theta) = J_0(x) + 2[J_2(x)\cos 2\theta + J_4(x)\cos 4\theta + \ldots\ldots]} \qquad \ldots(3)$$

$$\mathbf{\sin(x\sin\theta) = 2[J_1(x)\sin\theta + J_3(x)\sin 3\theta + \ldots\ldots]} \qquad \ldots(4)$$

These are known as **Jacobi series.**

Multiplying both sides of (3) by $\cos n\theta$ and integrating w.r.t. θ between the limits 0 and π (when n is odd, all terms on the RHS vanish; when n is even, all terms on the RHS except the one containing $\cos n\theta$ vanish), we get

$$\int_0^{\pi} \cos(x\sin\theta)\cos n\theta\, d\theta = \begin{cases} 0, & \text{when } n \text{ is odd} \\ \pi J_n(x), & \text{when } n \text{ is even} \end{cases}$$

Similarly, multiplying (4) by sin $n\theta$ and integrating w.r.t. θ between the limits 0 and π, we get

$$\int_0^{\pi} \sin(x \sin\theta) \sin n\theta \, d\theta = \begin{cases} \pi J_n(x), & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even} \end{cases}$$

Adding, we get $\int_0^{\pi} [\cos(x\sin\theta)\cos n\theta + \sin(x\sin\theta)\sin n\theta]\, d\theta = \pi J_n(x)$

$\Rightarrow$ $$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\, d\theta \quad \text{for all integral values of } n.$$

SOLVED EXAMPLES

Example 21. *Use Jacobi series to prove that*

$$[J_0(x)]^2 + 2[J_1(x)]^2 + 2[J_2(x)]^2 + 2[J_3(x)]^2 + \cdots\cdots = 1.$$

Sol. The Jacobi series are

$$J_0(x) + 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta + \cdots\cdots = \cos(x\sin\theta) \quad ...(1)$$

and $$2J_1(x)\sin\theta + 2J_3(x)\sin 3\theta + \cdots\cdots = \sin(x\sin\theta) \quad ...(2)$$

Squaring (1) and (2) and integrating w.r.t. θ between the limits 0 and π, and remembering that if m, n are integers then

$$\int_0^{\pi} \cos^2 n\theta\, d\theta = \int_0^{\pi} \sin^2 n\theta\, d\theta = \frac{\pi}{2}$$

and $$\int_0^{\pi} \cos m\theta \cos n\theta\, d\theta = \int_0^{\pi} \sin m\theta \sin n\theta\, d\theta = 0,\ m \neq n, \text{ we get}$$

$$[J_0(x)]^2\pi + 2[J_2(x)]^2\pi + 2[J_4(x)]^2\pi + \cdots\cdots = \int_0^{\pi} \cos^2(x\sin\theta)\, d\theta$$

$$2[J_1(x)]^2\pi + 2[J_3(x)]^2\pi + \cdots\cdots = \int_0^{\pi} \sin^2(x\sin\theta)\, d\theta$$

Adding, we have $\pi\{[J_0(x)]^2 + 2[J_1(x)]^2 + 2[J_2(x)]^2 + \cdots\cdots\}$

$$= \int_0^{\pi} [\cos^2(x\sin\theta) + \sin^2(x\sin\theta)]\, d\theta = \int_0^{\pi} d\theta = \pi\ .$$

$\therefore$ $$[J_0(x)]^2 + 2[J_1(x)]^2 + 2[J_2(x)]^2 + \cdots\cdots = 1.$$

Example 22. *Prove that:* $J_0(x) = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} \cos(x\cos\phi)\, d\phi.$

Sol. We know that

$$e^{x\cdot\frac{1}{2}\left(z - \frac{1}{z}\right)} = J_0 + \left(z - \frac{1}{z}\right)J_1 + \left(z^2 + \frac{1}{z^2}\right)J_2 + \left(z^3 - \frac{1}{z^3}\right)J_3 + \cdots \quad ...(1)$$

Putting $z = e^{i\theta}$ so that $\dfrac{1}{z} = e^{-i\theta}$

and $$z + \frac{1}{z} = 2\cos\theta\ ; \quad z - \frac{1}{z} = 2i\sin\theta,$$

NOTES

Eqn. (1) becomes,

$$e^{ix \sin\theta} = J_0 + (2i \sin\theta) J_1 + (2\cos 2\theta) J_2 + (2i \sin 3\theta) J_3 + (2\cos 4\theta) J_4 + \cdots$$

Equating real parts,

$$\cos(x \sin\theta) = J_0 + 2\cos 2\theta \, J_2 + 2\cos 4\theta \, J_4 + \cdots \qquad \ldots(2)$$

Putting $\theta = \dfrac{\pi}{2} + \phi$ in (2), we get

$$\cos(x\cos\phi) = J_0 + 2\cos(\pi + 2\phi) J_2 + 2\cos 4\left(\frac{\pi}{2} + \phi\right) J_4 + \cdots$$

$$= J_0 + (-2\cos 2\phi) J_2 + (2\cos 4\phi) J_4 + \cdots$$

$$\therefore \quad \int_0^\pi \cos(x\cos\phi)\, d\phi = J_0 \int_0^\pi d\phi - 2J_2 \int_0^\pi \cos 2\phi \, d\phi + 2J_4 \int_0^\pi \cos 4\phi \, d\phi - \cdots = \pi J_0$$

$$\therefore \quad J_0 = \frac{1}{\pi} \int_0^\pi \cos(x\cos\phi)\, d\phi .$$

EXERCISE A

1. Show that:

(*i*) $J_{1/2}(x) = J_{-1/2}(x) \cot x$ (*ii*) $J_{-5/2}(x) = \sqrt{\dfrac{2}{\pi x}}\left[\dfrac{3}{x}\sin x + \left(\dfrac{3 - x^2}{x^2}\right)\cos x\right]$

(*iii*) $[J_{1/2}(x)]^2 + [J_{-1/2}(x)]^2 = \dfrac{2}{\pi x}$

(*iv*) $J_{7/2}(x) = \sqrt{\dfrac{2}{\pi x}}\left[\left(\dfrac{15 - 6x^2}{x^3}\right)\sin x - \left(\dfrac{15}{x^2} - 1\right)\cos x\right]$.

2. Show that:

(*i*) $J_0'' = \dfrac{1}{2}(J_2 - J_0)$ (*ii*) $J_2 = J_0'' - x^{-1} J_0'$

(*iii*) $\dfrac{J_2}{J_1} = \dfrac{1}{x} - \dfrac{J_0''}{J_0'}$ (*iv*) $J_1''(x) = -J_1(x) + \dfrac{1}{x} J_2(x)$.

3. Prove that:

(*i*) $\dfrac{d}{dx}[x^n J_n(ax)] = ax^n J_{n-1}(ax)$ (*ii*) $\dfrac{d}{dx}[J_n^2(x)] = \dfrac{x}{2n}[J^2{}_{n-1}(x) - J^2{}_{n+1}(x)]$.

4. Prove that:

(*i*) $\int J_0(x) J_1(x)\, dx = -\dfrac{1}{2} J_0^2(x)$ (*ii*) $\int x^2 J_0(x) J_1(x)\, dx = \dfrac{1}{2} x^2 J_1^2(x)$

(*iii*) $\int \dfrac{J_4(x)}{x} dx = -\dfrac{1}{x} J_3(x) - \dfrac{2}{x^2} J_2(x)$ (*iv*) $\int J_5(x)\, dx = -J_4(x) - \dfrac{4}{x} J_3(x) - \dfrac{8}{x^2} J_2(x)$

(*v*) $\int x^3 J_0(x)\, dx = x^3 J_1(x) - 2x^2 J_2(x)$

(*vi*) $\int_0^\alpha x J_0(\lambda x)\, dx = \dfrac{\alpha}{\lambda} J_1(\alpha\lambda)$.

5. Prove that:

(*i*) $J_n(x) = \dfrac{1}{2\pi}\int_0^{2\pi} \cos(x\sin\theta - n\theta)\, d\theta$

(*ii*) $\cos x = J_0 - 2J_2 + 2J_4 - \cdots$; $\sin x = 2J_1 - 2J_3 + 2J_5 - \cdots$

[**Hint:** Put $\theta = \pi/2$ in Jacobi series] (*iii*) $\cos(x\cos\theta) = J_0 - 2J_2\cos 2\theta + 2J_4\cos 4\theta - \cdots\cdots$

$\sin(x\cos\theta) = 2J_1\cos\theta - 2J_3\cos 3\theta + 2J_5\cos 5\theta - \cdots\cdots$

[**Hint:** Replace θ by $\left(\frac{\pi}{2}-\theta\right)$ in Jacobi series]

(*iv*) $J_0(x) = \frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta)\,d\theta = \frac{2}{\pi}\int_0^{\pi/2}\cos(x\sin\theta)\,d\theta = \frac{2}{\pi}\int_0^{\pi/2}\cos(x\cos\theta)\,d\theta$

(*v*) $J_0 + 2J_2 + 2J_4 + 2J_6 + \ldots\ldots = 1$ (*vi*) $\int_0^{\pi/2}\sqrt{\pi x}\,J_{1/2}(2x)\,dx = 1.$

6. Show that Bessel's function $J_n(x)$ is an even function when n is even and is an odd function when n is odd. [**Hint:** $J_n(-x) = (-1)^n J_n(x)$]

7. (*i*) Express $J_6(x)$ in terms of $J_0(x)$ and $J_1(x)$.

(*ii*) Express $J_5(x)$ in terms of $J_0(x)$ and $J_1(x)$.

(*iii*) Show that $J_3(x) = \left(\frac{8}{x^2}-1\right)J_1(x) - \frac{4}{x}J_0(x)$.

8. Show that:

(*i*) $\int x^2 J_1(x)\,dx = 2xJ_1(x) - x^2J_0(x) + c$

(*ii*) $\int x^3 J_3(x)\,dx = -x^3J_2(x) - 5x^2J_1(x) - 15xJ_0(x) + 15\int J_0(x)\,dx$.

9. Evaluate: $\int x^4 J_1(x)\,dx$.

Answers

7. (*i*) $J_6(x) = \left(\frac{3840}{x^5} - \frac{768}{x^3} + \frac{18}{x}\right)J_1(x) + \left(\frac{144}{x^2} - \frac{1920}{x^4} - 1\right)J_0(x)$.

(*ii*) $J_5(x) = \left(\frac{384}{x^4} - \frac{72}{x^2} - 1\right)J_1(x) + \left(\frac{12}{x} - \frac{192}{x^3}\right)J_0(x)$.

9. $(8x^2 - x^4)J_0(x) + (4x^3 - 16x)J_1(x)$.

EQUATIONS REDUCIBLE TO BESSEL'S EQUATION

A number of second order differential equations with variable coefficients can be reduced to Bessel's equation by a suitable transformation and, hence, can be solved in terms of Bessel functions.

Consider the differential equation

$$x^2\frac{d^2y}{dx^2} + (1-2\alpha)x\frac{dy}{dx} + [\beta^2\gamma^2x^{2\gamma} + (\alpha^2 - n^2\gamma^2)]y = 0 \qquad \ldots(1)$$

where α, β, γ and n are constants.

Putting $X = \beta x^{\gamma}$ and $Y = x^{-\alpha}y$, equation (1) reduces to

$$X^2\frac{d^2Y}{dX^2} + X\frac{dY}{dX} + (X^2 - n^2)Y = 0 \qquad \ldots(2)$$

which is Bessel's equation.

When n is not an integer, the solution of (2) is

$$\boxed{Y = c_1 J_n (X) + c_2 J_{-n} (X)}$$

and hence, the solution of (2) is

$$x^{-\alpha} y = c_1 J_n (\beta x^\gamma) + c_2 J_{-n} (\beta x^\gamma)$$

or
$$\boxed{y = x^\alpha [c_1 J_n (\beta x^\gamma) + c_2 J_{-n}(\beta x^\gamma)]}$$

When n is an integer, the solution of (2) is

$$Y = c_1 J_n (X) + c_2 Y_n (X)$$

and hence, the solution of (2) is

$$\boxed{y = x^\alpha [c_1 J_n (\beta x^\gamma) + c_2 Y_n (\beta x^\gamma)].}$$

Equation (1) is a general form of Bessel's equation with α, β, γ and n as parameters. Comparing the given equation with (1), we get specific values of the parameters and hence the solution.

SOLVED EXAMPLES

Example 23. *Solve the following differential equations in terms of Bessel functions:*

(*i*) $y'' - \frac{2}{x} y' + 4\left(x^2 - \frac{1}{x^2}\right)y = 0$ (*ii*) $xy'' - 3y' + xy = 0.$

Sol. (*i*) The given equation is $x^2y'' - 2xy' + (4x^4 - 4)y = 0$

Comparing with the general form, we get

$$1 - 2\alpha = -2,\ \beta^2\gamma^2 = 4,\ 2\gamma = 4,\ \alpha^2 - n^2\gamma^2 = -4$$

i.e.,
$$\alpha = \tfrac{3}{2},\ \beta = 1,\ \gamma = 2,\ n = \tfrac{5}{4}.$$

Here n is not an integer and the solution is

$$y = x^{3/2} [c_1 J_{5/4} (x^2) + c_2 J_{-5/4} (x^2)]$$

(*ii*) Multiplying by x, the given equation becomes

$$x^2y'' - 3xy' + x^2y = 0$$

Comparing with the general form, we get

$$1 - 2\alpha = -3,\ \beta^2\gamma^2 = 1,\ 2\gamma = 2,\ \alpha^2 - n^2\gamma^2 = 0$$

i.e.,
$$\alpha = 2,\ \beta = \gamma = 1,\ n = 2$$

Here n is integer and the solution is $y = x^2[c_1 J_2 (x) + c_2 Y_2 (x)]$.

Example 24. *Obtain in terms of Bessel functions, the solution of differential equation*

$$\frac{d^2y}{dx^2} + \left(9x - \frac{20}{x^2}\right)y = 0.$$

Sol. The given equation on multiplying by x^2 is

$$x^2 \frac{d^2y}{dx^2} + (9x^3 - 20)y = 0$$

Comparing this with the standard transformed equation

$$x^2y'' + (1 - 2\alpha)\, xy' + \{\beta^2\gamma^2\, x^{2\gamma} + (\alpha^2 - n^2\gamma^2)\}\, y = 0,$$

we get $\quad 1 - 2\alpha = 0,\ \beta^2\gamma^2 = 9,\ 2\gamma = 3, \quad \text{and} \quad \alpha^2 - n^2\gamma^2 = -\,20$

This gives, $\quad \alpha = \dfrac{1}{2},\ \gamma = \dfrac{3}{2},\ \beta = 2,\ n = 3$

Here n is an integer.

Hence the solution is $\quad y = \sqrt{x}\ [c_1 J_3(2x^{3/2}) + c_2 Y_3(2x^{3/2})]$

Example 25. *Solve the differential equation*

$$y'' + \frac{y'}{x} + 4\left(x^2 - \frac{n^2}{x^2}\right) y = 0 \quad \textit{in terms of Bessel's functions.}$$

Sol. The given equation is

$$x^2y'' + xy' + 4\,(x^4 - n^2)\, y = 0 \qquad \ldots(1)$$

Comparing with the general form, we get

$$1 - 2\alpha = 1 \qquad \ldots(2)$$

$$\beta^2\gamma^2 = 4 \qquad \ldots(3)$$

$$2\gamma = 4 \qquad \ldots(4)$$

$$\alpha^2 - m^2\gamma^2 = -\,4n^2 \qquad \ldots(5)$$

From (2), $\quad \alpha = 0$

From (4), $\quad \gamma = 2$

From (3), $\quad \beta^2 = 1 \quad \Rightarrow \quad \beta = 1$

From (5), $\quad 0 - m^2\,(4) = -\,4n^2 \quad \Rightarrow \quad m = n$

when n is not an integer, solution to (1) is

$$y = x^{\alpha}\,[c_1\, J_m\,(\beta x^{\gamma}) + c_2 J_{-m}\,(\beta x^{\gamma})]$$

$$= x^0\,[c_1 J_n\,(x^2) + c_2 J_{-n}\,(x^2)] = c_1 J_n(x^2) + c_2 J_{-n}\,(x^2)$$

when n is an integer, solution to (1) is

$$y = x^{\alpha}\,[c_1\, J_m\,(\beta x^{\gamma}) + c_2 \gamma_m\,(\beta x^{\gamma})]$$

$$= x^0\ [c_1 J_n\,(x^2) + c_2 Y_n\,(x^2)] = c_1 J_n\,(x^2) + c_2 Y_n\,(x^2).$$

MODIFIED BESSEL'S EQUATION

The differential equation $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - (x^2 + n^2)\, y = 0 \qquad \ldots(1)$

is called modified Bessel's equation of order n.

Equation (1) can be re-written as $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (i^2x^2 - n^2)\, y = 0$

When n is not an integer, its solution is given by $y = c_1 J_n\,(ix) + c_2 J_{-n}\,(ix)$

Now, $J_n\,(ix) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma\,(n+k+1)} \left(\frac{ix}{2}\right)^{n+2k}$

$$= i^n \sum_{r=0}^{\infty} \frac{1}{k!\,\Gamma\,(n+k+1)} \left(\frac{x}{2}\right)^{n+2k} \qquad [\because \ (-1)^k\,(i)^{2k} = i^{2k}\,.\,i^{2k} = i^{4k} = 1]$$

The series $\sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(n+k+1)}\left(\frac{x}{2}\right)^{n+2k}$ is a real function with all terms positive.

It is denoted by $I_n(x)$ and is called the **modified Bessel function of the first kind of order *n*.**

Thus, $I_n(x) = i^{-n} J_n(ix)$

Since i^{-n} is a constant, $I_n(x)$ is also a solution of (1).

If n is not an integer, a second independent solution of (1) is $I_{-n}(x)$,

where $I_{-n}(x) = \sum_{r=0}^{\infty} \frac{1}{k!\,\Gamma(-n+k+1)}\left(\frac{x}{2}\right)^{-n+2k}$

Thus, if n is not an integer, the complete solution of (1) is given by

$$y = c_1 I_n(x) + c_2 I_{-n}(x)$$

If n is a non-zero integer, a second independent solution of (1) is given by

$$K_n(x) = \frac{\pi/2}{\sin n\pi}\,[I_{-n}(x) - J_n(x)]$$

and is called **modified Bessel function of second kind of order *n*.**

In this case, the complete solution of (1) is given by

$$y = c_1 I_n(x) + c_2 K_n(x).$$

BER AND BEI FUNCTIONS

Consider the differential equation $x\frac{d^2y}{dx^2} + \frac{dy}{dx} - ixy = 0$ ...(1)

Comparing it with equation (1) of Art. 4.7, we have

$$\alpha = 0,\ n = 0,\ \gamma = 1 \text{ and } \beta^2 = -i \quad \text{or} \quad \beta^2 = i^3 \quad \text{so that} \quad \beta = i^{3/2}.$$

Hence a solution of (1) is given by $J_0(i^{3/2}x)$

Replacing x by $i^{3/2}x$ in the series for $J_0(x)$, we have

$$J_0(i^{3/2}x) = 1 - \frac{i^3x^2}{2^2} + \frac{i^6x^4}{(2!)^2\cdot 2^4} - \frac{i^9x^6}{(3!)^2\cdot 2^6} + \frac{i^{12}x^8}{(4!)^2\,.\,2^8} - \ldots\ldots$$

$$= \left[1 - \frac{x^4}{2^2\cdot 4^2} + \frac{x^8}{2^2\cdot 4^4\cdot 6^2\cdot 8^2} - \ldots..\right]$$

$$+\, i\left[\frac{x^2}{2^2} - \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \frac{x^{10}}{2^2\cdot 4^2\cdot 6^2\cdot 8^2\cdot 10^2} - \ldots\ldots.\right]$$

Thus $J_0(i^{2/3}x)$ is a complex function for real values of x. The real and the imaginary parts are denoted by ber (x) (Bessel-real) and bei (x) (Bessel-imaginary) respectively. Thus

$$\text{ber}(x) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^{4k}}{2^2\cdot 4^2\cdot 6^2 \ldots\ldots. (4k)^2}$$

and
$$\text{bei}(x) = 1 - \sum_{k=1}^{\infty} \frac{(-1)^k x^{4k-2}}{2^2\cdot 4^2\cdot 6^2 \ldots\ldots. (4k-2)^2}$$

Hence a solution of (1) is $y = J_0(i^{3/2}x) = \text{ber}(x) + i\,\text{bei}(x)$.

Example 26. *Show that:*

(a) $\frac{d}{dx}[x\ ber'(x)] = -x\ bei\ (x)$ *(b)* $\frac{d}{dx}[x\ bei'(x)] = x\ ber\ (x)$.

Sol. We know that:

$$\text{ber}\ (x) = 1 - \frac{x^4}{2^2 \cdot 4^2} + \frac{x^8}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} - \ldots\ldots.$$

$$\text{bei}\ (x) = \frac{x^2}{2^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \frac{x^{10}}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2 \cdot 10^2} - \ldots\ldots.$$

(*a*) Now, $$\text{ber}'\ (x) = -\frac{x^3}{2^2 \cdot 4} + \frac{x^7}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} - \ldots\ldots..$$

$$\therefore \quad x\ \text{ber}'\ (x) = -\frac{x^4}{2^2 \cdot 4} + \frac{x^8}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} - \ldots\ldots..$$

$$\frac{d}{dx}[x\ \text{ber}'\ (x)] = -\frac{x^3}{2^2} + \frac{x^7}{2^2 \cdot 4^2 \cdot 6^2} - \ldots\ldots..$$

$$= -x\left(\frac{x^2}{2^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots\ldots..\right) = -x\ \text{bei}\ (x).$$

(*b*) Also, $$\text{bei}'\ (x) = \frac{x}{2} - \frac{x^5}{2^2 \cdot 4^2 \cdot 6} + \frac{x^9}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2 \cdot 10} - \ldots\ldots..$$

$$\therefore \quad x\ \text{bei}'\ (x) = \frac{x^2}{2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6} + \frac{x^{10}}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2 \cdot 10} - \ldots\ldots..$$

$$\frac{d}{dx}[x\ \text{bei}'\ (x)] = x - \frac{x^5}{2^2 \cdot 4^2} + \frac{x^9}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} - \ldots\ldots..$$

$$= x\left(1 - \frac{x^4}{2^2 \cdot 4^2} + \frac{x^8}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} - \ldots\ldots..\right) = x\ \text{ber}\ (x).$$

ORTHOGONALITY OF BESSEL FUNCTIONS

If α and β are the roots of $J_n(x) = 0$, then

$$\int_0^1 x\, J_n(\alpha x) \,.\, J_n(\beta x)\, dx = \begin{cases} 0, & \text{when } \alpha \neq \beta \\ \frac{1}{2} J^2_{n+1}(\alpha), & \text{when } \alpha = \beta \end{cases}$$

Consider the Bessel's equations

$$x^2u'' + xu' + (\alpha^2x^2 - n^2)u = 0 \qquad \ldots(1)$$

and $$x^2v'' + xv' + (\beta^2x^2 - n^2)\, v = 0 \qquad \ldots(2)$$

Their solutions are $u = J_n(\alpha x)$ and $v = J_n(\beta x)$ respectively.

Multiplying (1) by $\frac{v}{x}$ and (2) by $\frac{u}{x}$ and subtracting, we get

$$x(u''v - uv'') + (u'v - uv') + (\alpha^2 - \beta^2)\, xuv = 0$$

or $$\frac{d}{dx}[x(u'v - uv')] = (\beta^2 - \alpha^2)\, xuv$$

NOTES

Integrating both sides w.r.t. x between the limits 0 and 1, we get

$$(\beta^2 - \alpha^2) \int_0^1 xuv\, dx = \Big[x\,(u'v - uv') \Big]_0^1 = \Big[u'v - uv' \Big]_{x=1} \qquad ...(3)$$

Since, $\qquad u = J_n(\alpha x)$

$\therefore \qquad u' = \dfrac{d}{dx}[J_n(\alpha x)] = \dfrac{d}{d(\alpha x)}(J_n(\alpha x)] \,.\, \dfrac{d(\alpha x)}{dx} = \alpha J_n{}'(\alpha x)$

Similarly, $\qquad v = J_n(\beta x) \;\Rightarrow\; v' = \beta J_n{}'(\beta x)$

Substituting for u, v, u' and v' in (3), we get

$$\int_0^1 xJ_n(\alpha x)\, J_n(\beta x)\, dx = \frac{\alpha J_n'(\alpha)\, J_n(\beta) - \beta J_n(\alpha)\, J_n'(\beta)}{\beta^2 - \alpha^2} \qquad ...(4)$$

If α and β are *distinct* roots of $J_n(x) = 0$, then $J_n(\alpha) = 0$ and $J_n(\beta) = 0$.

Hence, from (4), we have

$$\int_0^1 xJ_n(\alpha x)\, J_n(\beta x)\, dx = 0$$

However, if $\alpha = \beta$, the value of the integral is $\dfrac{0}{0}$, which is indeterminate.

To evaluate the integral, we assume that α is a root of $J_n(x) = 0$ so that $J_n(\alpha) = 0$ and β is a variable approaching α. Thus, from (4), we have

$$\operatorname*{Lt}_{\beta \to \alpha} \int_0^1 xJ_n(\alpha x)\, J_n(\beta x)\, dx = \operatorname*{Lt}_{\beta \to \alpha} \frac{\alpha J_n'(\alpha)\, J_n(\beta)}{\beta^2 - \alpha^2}$$

or

$$\int_0^1 xJ_n^2(\alpha x)\, dx = \operatorname*{Lt}_{\beta \to \alpha} \frac{\alpha J_n'(\alpha)\, J_n(\beta)}{2\beta} \qquad \text{[by L-Hospital's rule]}$$

$$= \frac{1}{2}\,[J_n'(\alpha)]^2 \qquad ...(5)$$

But $\qquad xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$

$\therefore \qquad xJ_n'(\alpha) = nJ_n(\alpha) - \alpha J_{n+1}(\alpha) = -\alpha J_{n+1}(\alpha)$, since $J_n(\alpha) = 0$

$\Rightarrow \qquad J_n{}'(\alpha) = -J_{n+1}(\alpha)$

Hence, from (5), we get $\displaystyle\int_0^1 xJ_n{}^2(\alpha x)\, dx = \frac{1}{2} J^2{}_{n+1}(\alpha)$.

Note. If the interval is from 0 to a, it can be shown that

$$\int_0^a xJ_n^2(\alpha x)\, dx = \frac{a^2}{2} J_{n+1}^2(\alpha a), \quad \text{where } \alpha \text{ is a root of } J_n(\alpha a) = 0.$$

FOURIER-BESSEL EXPANSION OF $f(x)$

From the orthogonal property of Bessel functions, we can expand a function $f(x)$ in Fourier-Bessel series in the range 0 to a.

Let $\quad f(x) = c_1 J_n(\lambda_1 x) + c_2 J_n(\lambda_2 x) + \cdots\cdots + c_n J_n(\lambda_n x) + \cdots\cdots = \displaystyle\sum_{i=1}^{\infty} c_1 J_n(\lambda_i x) \qquad ...(1)$

where $\lambda_1, \lambda_2, \ldots\ldots$ are the roots of the equation $J_n(\lambda a) = 0$.

To determine c_i, we multiply both sides of (1) by $x\mathrm{J}_n(\lambda_i x)$ and integrate w.r.t. x between the limits 0 to a. From the orthogonal property of Bessel functions, all integrals on the right hand side will vanish except the one containing c_i and we have

$$\int_0^a xf(x)\,\mathrm{J}_n(\lambda_i x)\,dx = c_i \int_0^a x\mathrm{J}_n^{\,2}(\lambda_i x)\,dx = c_i \cdot \frac{a^2}{2}\mathrm{J}^2_{n+1}(\lambda_i a)$$

$$\therefore \qquad c_i = \frac{2}{a^2\mathrm{J}^2_{n+1}(\lambda_{ia})}\int_0^a xf(x)\,\mathrm{J}_n(\lambda_i x)\,dx$$

Putting $i = 1, 2, 3, ...$ we can find $c_1, c_2, c_3, ...$ and hence the function $f(x)$.

SOLVED EXAMPLES

Example 27. *If $\alpha_1, \alpha_2, ..., \alpha_n$ are the positive roots of $J_0(x) = 0$, prove that*

$$\frac{1}{2} = \sum_{n=1}^{\infty} \frac{J_0(\alpha_n x)}{\alpha_n J_1(\alpha_n)}.$$

Sol. We know that if $f(x) = \sum_{i=1}^{\infty} c_i \mathrm{J}_n(\alpha_i x)$ ...(1)

then $$c_i = \frac{2}{a^2\mathrm{J}^2_{n+1}(\alpha_i a)}\int_0^a xf(x)\,\mathrm{J}_n(\alpha_i x)\,dx$$

Taking $f(x) = 1$, $a = 1$ and $n = 0$, we get

$$c_i = \frac{2}{\mathrm{J}_1^{\,2}(\alpha_i)}\int_0^1 x\mathrm{J}_0(\alpha_i x)\,dx = \frac{2}{\mathrm{J}_1^{\,2}(\alpha_i)}\left[\frac{x\mathrm{J}_1(\alpha_i x)}{\alpha_i}\right]_0^1 = \frac{2}{\alpha_i\,\mathrm{J}_1(\alpha_i)}$$

$\therefore$ From (1), we have $1 = \sum_{i=1}^{\infty} \frac{2}{\alpha_i\,\mathrm{J}_1(\alpha_i)}\mathrm{J}_0(\alpha_i x)$

or $$\frac{1}{2} = \sum_{n=1}^{\infty} \frac{\mathrm{J}_0(\alpha_n x)}{\alpha_n\,\mathrm{J}_1(\alpha_n)}.$$

Example 28. *Show that the Fourier-Bessel series in $J_2(\lambda_n x)$ for $f(x) = x^2$ $(0 < x < a)$, where $\lambda_n a$ are positive roots of $J_2(x) = 0$, is*

$$x^2 = 2a^2 \sum_{n=1}^{\infty} \frac{J_2(\lambda_n x)}{a\lambda_n J_3(\lambda_n a)}.$$

Sol. Let the Fourier-Bessel series representing $f(x) = x^2$ be given by

$$x^2 = \sum_{n=1}^{\infty} c_n \mathrm{J}_2(\lambda_n x)$$

Multiplying both sides by $x\mathrm{J}_2(\lambda_n x)$ and integrating w.r.t. x between the limits 0 to a, we get

$$\int_0^a x^3\mathrm{J}_2(\lambda_n x)\,dx = c_n \int_0^a x\mathrm{J}_2^2(\lambda_n x)\,dx$$

or $$\left[\frac{x^3\mathrm{J}_3(\lambda_n x)}{\lambda_n}\right]_0^a = c_n \cdot \frac{a^2}{2}\mathrm{J}_3^2(\lambda_n a)$$

NOTES

or $$\frac{a^3 J_3(\lambda_n a)}{\lambda_n} = c_n \cdot \frac{a^2}{2} J_3^2(\lambda_n a)$$

$\therefore$ $$c_n = \frac{2a^2}{a\lambda_n} \cdot \frac{1}{J_3(\lambda_n a)}$$

Hence, $$x^2 = 2a^2 \sum_{n=1}^{\infty} \frac{J_2(\lambda_n x)}{a\lambda_n J_3(\lambda_n a)}.$$

EXERCISE B

1. Solve the following differential equations in terms of Bessel functions:

(*i*) $xy'' + y = 0$

(*ii*) $xy'' - y' + 4x^3 y = 0$

(*iii*) $y'' + \frac{1}{x}y' + 4\left(1 - \frac{1}{x^2}\right) y = 0$

(*iv*) $y'' + \frac{1}{x}y' + \left(3 - \frac{1}{4x^2}\right) y = 0$

(*v*) $y'' + \left(9 - \frac{20}{x^2}\right) y = 0.$

(*vi*) $x^2 y'' - xy' + 4x^2 y = 0$

(*vii*) $y'' + \frac{1}{x}y' + \left(8 - \frac{1}{x^2}\right)y = 0$

(*viii*) $4y'' + 9xy = 0$

(*ix*) $xy'' + y' + \frac{1}{4}y = 0$

(*x*) $y'' + \frac{y'}{x} + \left(1 - \frac{1}{9x^2}\right)y = 0$

2. Expand $f(x) = 1$ over the interval $0 < x < 3$ in terms of the functions $J_0(\lambda_n x)$, where λ_n are determined by $J_0(3\lambda) = 0$.

3. Expand $f(x) = 4x - x^3$ over the interval (0, 2) in terms of Bessel functions of first kind of order one which satisfy the condition $[J_1(\lambda x)]_{x=2} = 0$.

4. If $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n, \ldots\ldots$ are the positive roots of $J_1(x) = 0$, prove that

(*i*) $x^2 = \frac{1}{2} + 4 \sum_{n=1}^{\infty} \frac{J_0(\alpha_n x)}{\alpha_n^2 J_0(\alpha_n)}$

(*ii*) $(1 - x^2)^2 = \frac{1}{3} - 64 \sum_{n=1}^{\infty} \frac{J_0(\alpha_n x)}{\alpha_n^2 J_0(\alpha_n)}.$

5. If a is the root of the equation $J_0(x) = 0$, show that

(*i*) $\int_0^1 J_1(ax)\, dx = \frac{1}{a}$

(*ii*) $\int_0^a J_1(x)\, dx = 1.$

Answers

1. (*i*) $y = \sqrt{x}\,[c_1 J_1(2\sqrt{x}) + c_2 Y_1(2\sqrt{x})]$

(*ii*) $y = x\,[c_1 J_{1/2}(x^2) + c_2 J_{-1/2}(x^2)]$

(*iii*) $y = c_1 J_2(2x) + c_2 Y_2(2x)$

(*iv*) $y = c_1 J_{1/2}(\sqrt{3}x) + c_2 J_{-1/2}(\sqrt{3}x)$

(*v*) $y = \sqrt{x}\,[c_1 J_{9/2}(3x) + c_2 J_{-9/2}(3x)]$

(*vi*) $y = x[c_1 J_1(2x) + c_2 Y_1(2x)]$

(*vii*) $y = c_1 J_2(4\sqrt{2x}) + c_2 Y_2(4\sqrt{2x})$

(*viii*) $y = x^{1/2}\,[c_1 J_{1/3}(x^{3/2}) + c_2 J_{-1/3}(x^{3/2})]$

(*ix*) $y = c_1 J_0(\sqrt{x}) + c_2 Y_0(\sqrt{x})$

(*x*) $y = c_1 J_{1/3}(x) + c_2 J_{-1/3}(x)$

2. $1 = \frac{2}{3} \sum_{n=1}^{\infty} \frac{J_0(\lambda_n x)}{\lambda_n J_1(3\lambda_n)}.$

3. $4x - x^3 = 8 \sum_{n=1}^{\infty} \frac{J_3(2\lambda_n)}{\lambda_n^2 J_2^2(2\lambda_n)} \cdot J_1(\lambda_n x).$